FOURIER SERIES

AND

BOUNDARY VALUE PROBLEMS

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BY

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PREFACE

This is an introductory treatment of Fourier series and their application to the solution of boundary value problems in the partial differential equations of physics and engineering. It is designed for students who have had an introductory course in ordinary differential equations and one semester of advanced calculus, or an equivalent preparation. The concepts from the field of physics which are involved here are kept on an elementary level. They are explained in the early part of the book, so that no previous preparation in this direction need be assumed.

The first objective of this book is to introduce the reader to the concept of orthogonal sets of functions and to the basic ideas of the use of such functions in representing arbitrary functions. The most prominent special case, that of representing an arbitrary function by its Fourier series, is given special attention. The Fourier integral representation and the representation of functions by series of Bessel functions and Legendre polynomials are also treated individually, but somewhat less fully. The material covered is intended to prepare the reader for the usual applications arising in the physical sciences and to furnish a sound background for those who wish to pursue the subject further.

The second objective is a thorough acquaintance with the classical process of solving boundary value problems in partial differential equations, with the aid of those expansions in series of orthogonal functions. The boundary value problems treated here consist of a variety of problems in heat conduction, vibration, and potential. Emphasis is placed on the formal method of obtaining the solutions of such problems. But attention is also given to the matters of fully establishing the results as solutions and of investigating their uniqueness, for the process cannot be properly presented without some consideration of these matters.

The book is intended to be both elementary and mathematically sound. It has been the author's experience that careful attention to the mathematical development, in contrast

to more formal procedures, contributes much to the student's interest as well as to his understanding of the subject, whether he is a student of pure or of applied mathematics. The few theorems that are stated here without proofs appear at the end of the discussion of the topics concerned, so they do not reflect upon the completeness of the earlier part of the development.

Illustrative examples are given whenever new processes are involved.

The problems form an essential part of such a book. A rather generous supply and wide variety will be found here. Answers are given to all but a few of the problems.

The chapters on Bessel functions and Legendre polynomials (Chaps. VIII and IX) are independent of each other, so that they can be taken up in either order. The continuity of the subject matter will not be interrupted by omitting the chapter on the uniqueness of solutions of boundary value problems (Chap. VII) or by omitting certain parts of other chapters.

This volume is a revision and extension of a planographed form developed by the author in a course given for many years to students of physics, engineering, and mathematics at the University of Michigan. It is to be followed soon by a more advanced book on further methods of solving boundary value problems.

The selection and presentation of the material for the present volume have been influenced by the works of a large number of authors, including Carslaw, Courant, Byerly, Bôcher, Riemann and Weber, Watson, Hobson, and several others.

To Dr. E. D. Rainville and Dr. R. C. F. Bartels the author wishes to express his gratitude for valuable suggestions and for their generous assistance with the reading of proof. In the preparation of the manuscript he has been faithfully assisted by his daughter, who did most of the typing, and by his wife and son.

RUEL V. CHURCHILL.

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CHAPTER I

INTRODUCTION

1. The Two Related Problems. We shall be concerned here with two general types of problems: (*a*) the expansion of an arbitrarily given function in an infinite series whose terms are certain prescribed functions and (*b*) boundary value problems in the partial differential equations of physics and engineering. These two problems are so closely related that there are many advantages, especially to those interested in applied mathematics, in an introductory treatment that deals with both of them together.

In fact an acquaintance with the expansion theory is necessary for the study of boundary value problems. The expansion problem can be treated independently. It is an interesting problem in pure mathematics, and its applications are not confined to boundary value problems. But it gains in unity and interest when presented as a problem arising in the solution of partial differential equations.

The series in the problem type (*a*) is a Fourier series when its terms are certain linear combinations of sines and cosines. Fourier encountered this expansion problem, and made the first extensive treatment of it, in his development of the mathematical theory of the conduction of heat in solids.* Before Fourier's work, however, the investigations of others, notably D. Bernoulli and Euler, on the vibrations of strings, columns of air, elastic rods, and membranes, and of Legendre and Laplace on the theory of gravitational potential, had led to expansion

* Fourier, "Théorie analytique de la chaleur," 1822. A translation of this book by Freeman appeared in 1878 under the title "The Analytical Theory of Heat."

problems of the kind treated by Fourier as well as the related problems of expanding functions in series of Bessel functions, Legendre polynomials, and spherical harmonic functions.

These physical problems which led the early investigators to the various expansions are all examples of boundary value problems in partial differential equations. Our plan of presentation here is in agreement with the historical development of the subject.

The expansion problem as presented here will stress the development of functions in Fourier series. But we shall also consider the related generalized Fourier development of an arbitrary function in series of orthogonal functions, including the important series of Bessel functions and Legendre polynomials.

2. Linear Differential Equations. An equation in a function of two or more variables and its partial derivatives is called a partial differential equation. The *order* of a partial differential equation, as in the case of an ordinary differential equation, is that of the highest ordered derivative appearing in it. Thus the equation

$$\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial y} = 3xy \tag{1}$$

is one of the second order.

A partial differential equation is *linear* if it is of the first degree in the unknown function and its derivatives. The equation

$$\frac{\partial^2 u}{\partial x^2} + xy^2\frac{\partial u}{\partial y} = 3xy \tag{2}$$

is linear; equation (1) is nonlinear. If the equation contains only terms of the first degree in the function and its derivatives, it is called a *linear homogeneous* equation. Equation (2) is nonhomogeneous, but the equation

$$\frac{\partial^2 u}{\partial x^2} + xy^2\frac{\partial u}{\partial y} = 0$$

is linear and homogeneous.

Thus the general linear partial differential equation of the second order, in two independent variables x and y, is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where the letters $A, B, \cdots, G$, represent functions of x and y. If F is identically zero, the equation is homogeneous.

The following theorem is sometimes referred to as the *principle of superposition* of solutions.

Theorem 1. *Any linear combination of two solutions of a linear homogeneous differential equation is again a solution.*

The proof for the ordinary equation

$$y'' + Py' + Qy = 0, \tag{3}$$

where P and Q may be functions of x, will show how the proof can be written for any linear homogeneous differential equation, ordinary or partial.

Let $y = y_1(x)$ and $y = y_2(x)$ be two solutions of equation (3). Then

$$y_1'' + Py_1' + Qy_1 = 0, \tag{4}$$

$$y_2'' + Py_2' + Qy_2 = 0. \tag{5}$$

It is to be shown that any linear combination of y_1 and y_2—namely, $Ay_1 + By_2$, where A and B are arbitrary constants—is a solution of equation (3). By multiplying equations (4) by A and (5) by B and adding, the equation

$$Ay_1'' + By_2'' + P(Ay_1' + By_2') + Q(Ay_1 + By_2) = 0$$

is obtained. This can be written

$$\frac{d^2}{dx^2}(Ay_1 + By_2) + P\frac{d}{dx}(Ay_1 + By_2) + Q(Ay_1 + By_2) = 0,$$

which is a statement that $Ay_1 + By_2$ is a solution of equation (3).

For an ordinary differential equation of order n, a solution containing n arbitrary constants is known as the general solution. But a partial differential equation of order n has in general a solution containing n arbitrary functions. These are functions of $k - 1$ variables, where k represents the number of independent variables in the equation. On those few occasions here where we consider such solutions, we shall refer to them as "general solutions" of the partial differential equations. But the collection of all possible solutions of a partial differential equation is not simple enough to be represented by just this "general solution" alone.*

* See, for instance, Courant and Hilbert, "Methoden der mathematischen Physik," Vol. 2, Chap. I; or Forsyth, "Theory of Differential Equations," Vols. 5 and 6.

Consider, for example, the simple partial differential equation in the function $u(x, y)$:

$$\frac{\partial u}{\partial x} = 0.$$

According to the definition of the partial derivative, the solution is

$$u = f(y),$$

where $f(y)$ is an arbitrary function. Similarly, when the equation

$$\frac{\partial^2 u}{\partial x^2} = 0$$

is written $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 0$, its general solution is seen to be

$$u = xf(y) + g(y),$$

where $f(y)$ and $g(y)$ are arbitrary functions.

PROBLEMS

1. Prove Theorem 1 for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

2. Prove Theorem 1 for the heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Note that k may be a function of x, y, z, and t here.

3. Show by means of examples that the statement in Theorem 1 is not always true when the differential equation is nonhomogeneous.

4. Show that $y = f(x + at)$ and $y = g(x - at)$ satisfy the simple wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

where a is a constant and f and g are arbitrary functions, and hence that a general solution of that equation is

$$y = f(x + at) + g(x - at).$$

5. Show that $e^{-n^2 t} \sin nx$ is a solution of the simple heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

If $A_1, A_2, \cdots, A_N$ are constants, show that the function

$$u = \sum_{n=1}^{N} A_n e^{-n^2 t} \sin nx$$

is a solution having the value zero at $x = 0$ and $x = \pi$, for all t.

3. Infinite Series of Solutions. Let u_n $(n = 1, 2, 3, \cdots)$ be an infinite set of functions of any number of variables such that the series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

converges to a function u. If the series of derivatives of u_n, with respect to one of the variables, converges to the same derivative of u, then the first series is said to be *termwise differentiable* with respect to that variable.

Theorem 2. *If each of the functions $u_1, u_2, \cdots, u_n, \cdots$, is a solution of a linear homogeneous differential equation, the function*

$$u = \sum_{1}^{\infty} u_n$$

is also a solution provided this infinite series converges and is termwise differentiable as far as those derivatives which appear in the differential equation are concerned.

Consider the proof for the differential equation

$$\frac{\partial^2 u}{\partial x^2} + p \frac{\partial^2 u}{\partial x\, \partial t} + qu = 0, \tag{1}$$

where p and q may be functions of x and t. Let each of the functions $u_n(x, t)$ $(n = 1, 2, \cdots)$ satisfy equation (1). The series

$$\sum_{1}^{\infty} u_n(x, t)$$

is assumed to be convergent and termwise differentiable; hence if $u(x, t)$ represents its sum, then

$$\frac{\partial u}{\partial x} \qquad \sum_{1} \partial x, \qquad \overline{\partial x^2} \qquad \sum_{1} \partial^2 u_n \qquad \frac{\partial^2 u}{\partial x\, \partial t} \qquad \partial x\, \partial t$$

Substituting these, the left-hand member of equation (1) becomes

$$\sum_{1}^{\infty} \frac{\partial^2 u_n}{\partial x^2} + p \sum_{1}^{\infty} \frac{\partial^2 u_n}{\partial x\, \partial t} + q \sum_{1}^{\infty} u_n, \tag{2}$$

and if this quantity vanishes, the theorem is true. Now expression (2) can be written

$$\sum_{1}^{\infty}\left(\frac{\partial^2 u_n}{\partial x^2} + p\,\frac{\partial^2 u_n}{\partial x\,\partial t} + q u_n\right),$$

since the series obtained by adding three convergent series term by term converges to the sum of the three functions represented by those series. Since u_n is a solution of equation (1),

$$\frac{\partial^2 u_n}{\partial x^2} + p\,\frac{\partial^2 u_n}{\partial x\,\partial t} + q u_n = 0 \quad (n = 1, 2, \cdot \cdot \cdot),$$

and so expression (2) is equal to zero. Hence $u(x, t)$ satisfies equation (1).

This proof depends only upon the fact that the differential equation is linear and homogeneous. It can clearly be applied to any such equation regardless of its order or number of variables.

4. Boundary Value Problems. In applied problems in differential equations a solution which satisfies some specified conditions for given values of the independent variables is usually sought. These conditions are known as the boundary conditions. The differential equation together with these boundary conditions constitutes a boundary value problem. The student is familiar with such problems in ordinary differential equations. Consider, for example, the following problem.

A body moves along the x-axis under a force of attraction toward the origin proportional to its distance from the origin. If it is initially in the position $x = 0$ and its position one second later is $x = 1$, find its position $x(t)$ at every instant.

The displacement $x(t)$ must satisfy the conditions

(1) $$\frac{d^2x}{dt^2} = -k^2 x,$$

(2) $$x = 0 \text{ when } t = 0, \qquad x = 1 \text{ when } t = 1,$$

where k is a constant. The boundary value problem here consists of the equation (1) and the boundary conditions (2), which assign values to the function x at the extremities (or on the boundary) of the time interval from $t = 0$ to $t = 1$.

The general solution of equation (1) is

$$x = C_1 \cos kt + C_2 \sin kt.$$

According to the conditions (2), $C_1 = 0$ and $C_2 = 1/\sin k$, so the solution of the problem is

$$x = \frac{\sin kt}{\sin k}.$$

From this the initial velocity which makes $x = 1$ when $t = 1$ can be written

$$\frac{dx}{dt} \quad \frac{k}{\sin k} \qquad \text{when } t = 0.$$

This condition could have been used in place of either of the conditions (2) to form another boundary value problem with the same solution.

In general, the boundary conditions may contain conditions on the derivatives of the unknown function as well as on the function itself.

The method corresponding to the one just used can sometimes be applied in partial differential equations. Consider, for instance, the following boundary value problem in $u(x, y)$:

$$u(0, y) = y^2, \qquad u(1, y) = 1. \tag{4}$$

Here the values of u are prescribed on the boundary, consisting of the lines $x = 0$ and $x = 1$, of the infinite strip in the xy-plane between those lines.

The general solution of equation (3) is

$$u(x, y) = xf(y) + g(y),$$

where $f(y)$ and $g(y)$ are arbitrary functions. The conditions (4) require that

$$g(y) = y^2, \qquad f(y) + g(y) = 1, \tag{5}$$

so $f(y) = 1 - y^2$, and the solution of the problem is

$$u(x, y) = x(1 - y^2) + y^2.$$

But it is only in exceptional cases that problems in partial differential equations can be solved by the above method. The general solution of the partial differential equation usually cannot be found in any practical form. But even when a gen-

eral solution is known, the functional equations, corresponding to equations (5), which are given by the boundary conditions are often too difficult to solve. A more powerful method will be developed in the following chapters—a method of combining particular solutions with the aid of Theorems 1 and 2. It is, of course, limited to problems possessing a certain linear character.

The number and character of the boundary conditions which completely determine a solution of a partial differential equation depend upon the character of the equation. In the physical applications, however, the interpretation of the problem will indicate what boundary conditions are needed. If, after a solution of the problem is established, it is shown that only one solution is possible, the problem will have been shown to be completely stated as well as solved.

PROBLEMS

1. Solve the boundary value problem

$$\partial^2 u \qquad = 0; \qquad u(0, y) = y \qquad u(x, 0) = \sin x.$$

Ans. $u = y + \sin x.$

2. Solve the boundary value problem

$$\frac{\partial^2 u}{\partial x\, \partial y} = 2x; \qquad u(0, y) = 0, \qquad u(x, 0) = x^2.$$

Ans. $u = x^2 y + x^2.$

3. Solve Prob. 2 when the second boundary condition is replaced by the condition

$$\frac{\partial u(x, 0)}{\partial x} = x^2.$$

Ans. $u = x^2 y + \frac{1}{3}x^3.$

4. By substituting the new independent variables

$$\lambda = x + at, \qquad \mu = x - at,$$

show that the wave equation $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$ becomes

$$= 0,$$

and so derive the general solution of the wave equation (Prob. 4, Sec. 2).

5. Solve the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}; \qquad y(x, 0) = F(x), \qquad \frac{\partial y(x, 0)}{\partial t} = 0,$$

where $F(x)$ is a given function defined for all real x.

Ans. $y = \frac{1}{2}[F(x + at) + F(x - at)]$.

6. Solve Prob. 5 if the boundary conditions are replaced by

$$y(x, 0) = 0, \qquad \frac{\partial y(x, 0)}{\partial t} = G(x).$$

Also show that the solution under the more general conditions

$$y(x, 0) = F(x), \qquad \frac{\partial y(x, 0)}{\partial t}$$

is obtained by adding the solution just found to the solution of Prob. 5.

Ans. $y = \int_{x-at}^{x+}$

CHAPTER II

PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICS

5. Gravitational Potential. According to the universal law of gravitation, the force of attraction exerted by a particle of mass m at the point (x, y, z) upon a unit mass at (X, Y, Z) is directed along the line joining the two points, and its magnitude and sense are given by the equation

$$F = -\frac{km}{r^2}$$

where k is a positive constant and r is the distance between the two masses:

$$r = \sqrt{(X - x)^2 + (Y - y)^2 + (Z - z)^2}.$$

The positive sense is taken from the point (x, y, z), called Q, toward the point P (X, Y, Z).

The gravitational potential V at any point P due to the mass m at Q is defined to be the function

$$V = \frac{km}{r}.$$

So the derivative of this function is the force:

$$\frac{\partial V}{\partial r} = -\frac{km}{r^2} = F.$$

Let Q be fixed and consider V as a function of X, Y, and Z. It will now be shown that the directional derivative of the potential in any direction gives the projection of the force F in that direction.

First let the direction be parallel to the X-axis. Then

$$\frac{\partial V}{\partial X} \qquad \overline{\partial r}\ \overline{\partial X} \qquad km\, X - x = F \cos \alpha = .$$

where $\cos \alpha$ is the first direction cosine of the radius vector r,

and F_x is the projection of F on the X-axis. Similarly,

$$\frac{\partial V}{\partial Y} = F_y, \qquad \frac{\partial V}{\partial Z} = F_z. \tag{1}$$

Now if s is the directed distance along any line through P having the direction angles α', β', γ', the directional derivative of V can be written

$$\begin{aligned} \frac{\partial V}{\partial s} &= \frac{\partial V}{\partial X}\frac{\partial X}{\partial s} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial s} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial s} \\ &= F_x \cos \alpha' + F_y \cos \beta' + F_z \cos \gamma'. \end{aligned} \tag{2}$$

The last expression is the projection of the force in the direction of the line along which s is measured.

The extension to the potential and force due to a continuous distribution of mass is quite direct. The potential function due to a mass of density $\delta(x, y, z)$ distributed throughout a volume τ, at a point P not occupied by mass, is defined to be

$$V(X, Y, Z) = k \iiint_\tau \frac{\delta(x, y, z)\, dx\, dy\, dz}{[(X - x)^2 + (Y - y)^2 + (Z - z)^2]^{\frac{1}{2}}}. \tag{3}$$

This integral can be differentiated with respect to X, Y, or Z inside the integral. Thus

$$\frac{\partial V}{\partial X} = -k \iiint_\tau \frac{X - x}{r} \frac{\delta(x, y, z)}{r^2}\, dx\, dy\, dz. \tag{4}$$

This is the total component F_x of the gravitational forces exerted by all the elements of mass in τ upon a unit mass at P. Likewise the total components F_y and F_z satisfy relations (1), so that the directional derivative has the same form as in equation (2).

Hence the projection, along any direction, of the force exerted by a mass distribution upon a unit mass at (X, Y, Z) is given by the directional derivative, along that direction, of the potential function (3); that is,

A force which can be derived in this manner from a potential function is known as a conservative force.

Let s be the arc length along any curve joining two points (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2), at which $s = s_1$ and $s = s_2$, respectively. Then, according to formula (5),

$$\int_{s_1}^{s_2} \qquad\qquad - V(X_1$$

That is, the difference between the values of the potential V at two points represents the work done by the gravitational force upon a unit mass which is moved from one of these points to the other. The amount of work depends upon the positions of the points, but not upon the path along which the unit mass moves.

6. Laplace's Equation. The potential $V(X, Y, Z)$ due to any distribution of mass will now be seen to satisfy an important partial differential equation. Upon differentiating both members of equation (4), Sec. 5, with respect to X, we find that

$$\frac{\partial^2 V}{\partial X^2} = -k \int\int\int_\tau \left[\frac{1}{r^3} - \frac{3(X-x)^2}{r^5}\right] \delta\, dx\, dy\, dz.$$

Likewise

$$\frac{\partial^2 V}{\partial Y^2} = -k \int\int\int_\tau \left[\frac{1}{r^3} - \frac{3(Y-y)^2}{r^5}\right] \delta\, dx\, dy\, dz,$$

$$\frac{\partial^2 V}{dZ^2} = -k \int\int\int_\tau \left[\frac{1}{r^3} - \frac{3(Z-z)^2}{r^5}\right] \delta\, dx\, dy\, dz.$$

The sum of the terms inside the three brackets is zero; so

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} = 0.$$

This is *Laplace's equation.* It is often written

$$\nabla^2 V = 0,$$

where the *Laplacian operator* ∇^2, sometimes called "del squared," is defined as follows:

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}.$$

The same operator is present in several other important equations.

We have just shown that Laplace's equation is satisfied by the gravitational potential at points in space not occupied

by mass. It is satisfied as well by static electric or magnetic potential at points free from electric charges or magnetic poles, since the law of attraction or repulsion and the definition of the potential function in these cases are the same, except for constant factors, as in the case of gravitation.

Other important functions in the applications satisfy Laplace's equation. One of them is the velocity potential of the irrotational motion of an incompressible fluid, used in hydrodynamics and aerodynamics. Another is the steady temperature at points in a homogeneous solid; this will be shown further on in this chapter.

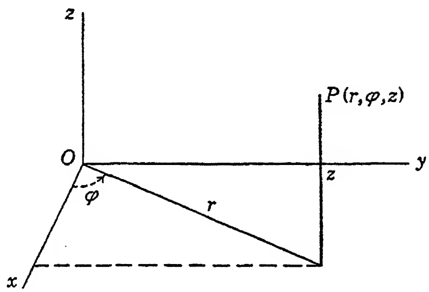

FIG. 1.

The gravitational potential at points occupied by mass of density δ can be shown to satisfy *Poisson's equation:*

a nonhomogeneous equation. The equations of Laplace and Poisson, like most of the important partial differential equations of physics, are linear and of the second order.

7. Cylindrical and Spherical Coordinates. Since cylindrical and spherical surfaces occur frequently in the boundary value problems of physics, it is important to have expressions for the Laplacian operator in terms of cylindrical and spherical coordinates.

The cylindrical coordinates (r, φ, z) determine a point P (Fig. 1) whose rectangular coordinates are

(1) $$x = r \cos \varphi, \qquad y = r \sin \varphi, \qquad z = z.$$

These relations can be written

(2) $$r = \qquad\qquad = \arctan \frac{y}{x}, \qquad = z,$$

provided it is observed that the quadrant of the angle φ is determined by the signs of x and y, not by the ratio y/x alone.

Let u be a function of r, φ, and z. In view of relations (2) it is also a function of x, y, z, and according to the formula for differentiating a composite function,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial u}{\partial \varphi}\frac{y}{x^2+y^2}.$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) - \frac{\partial u}{\partial \varphi}\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) + \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) - \frac{y}{r^2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \varphi}\right).$$

The last two indicated derivatives can be written

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2}\frac{x}{r} - \frac{\partial^2 u}{\partial r\,\partial \varphi}\frac{y}{r^2},$$
$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \varphi}\right) = \frac{\partial^2 u}{\partial \varphi\,\partial r}\frac{x}{r} - \frac{\partial^2 u}{\partial \varphi^2}\frac{y}{r^2}.$$

Substituting and simplifying, we find that

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{r^3}\frac{\partial u}{\partial r} + \frac{2xy}{r^4}\frac{\partial u}{\partial \varphi} + \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial r\,\partial \varphi} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial \varphi^2}.$$

Similarly, it is found that

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3}\frac{\partial u}{\partial r} - \frac{2xy}{r^4}\frac{\partial u}{\partial \varphi} + \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial r\,\partial \varphi} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial \varphi^2},$$

so that the *Laplacian* of u *in cylindrical coordinates* is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}. \tag{3}$$

It is simpler to transform the right-hand member of equation (3) into rectangular coordinates. This operation furnishes a verification of equation (3).

The spherical coordinates (r, φ, θ) of a point P (Fig. 2), also called polar coordinates, are related to the rectangular coordinates as follows:

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta. \tag{4}$$

The *Laplacian* of a function *u in spherical coordinates* is

(5) $$\sin^2 \theta \, \partial$$

The derivation or verification of this formula can be carried out in the same manner as that of the corresponding formula (3) for cylindrical coordinates. It is left as an exercise.

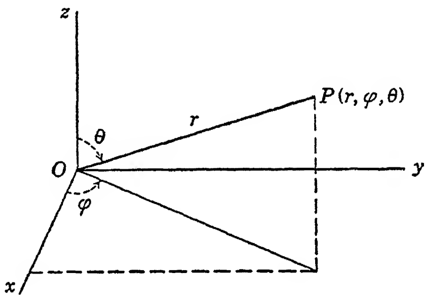

FIG. 2.

PROBLEMS

1. Derive the expression given above for $\partial^2 u/\partial y^2$ in cylindrical coordinates, and thus complete the derivation of formula (3).

2. Verify formula (3) by transforming its right-hand member into rectangular coordinates.

3. Verify formula (5) by transforming its right-hand member into rectangular coordinates.

4. Write the formulas which give the spherical coordinates in terms of x, y, z.

5. Derive formula (5) for the Laplacian in terms of spherical coordinates.

8. The Flux of Heat. Consider an infinite slab of homogeneous solid material bounded by the planes $x = 0$ and $x = L$. Let the faces $x = 0$ and $x = L$ be kept at fixed uniform temperatures u_1 and u_2, respectively. After the temperatures have become steady, the amount of heat per unit time which flows from the surface $x = 0$ to the surface $x = L$, per unit area, is

where the constant K is known as the thermal conductivity. This statement is essentially a definition of the conductivity K.

The time rate of flow of heat per unit area through a surface is called the *flux* of heat. For the flux F through any isothermal surface (a surface at uniform temperature), the natural extension of the above definition is

$$F = -K\frac{\partial u}{\partial n_0}. \tag{1}$$

Here u is the temperature as a function of position, n_0 is the distance measured along a directed normal to the isotherm, and the positive sense of the flux F is that of the normal. In formula (1) the conductivity K may be variable, and the solid nonhomogeneous.

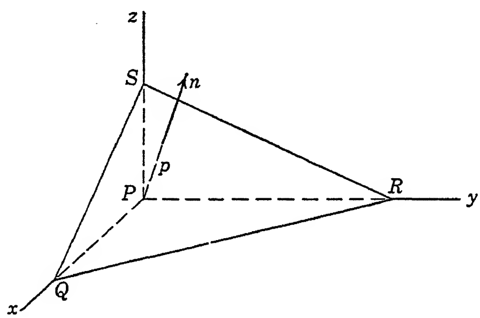

FIG. 3.

To indicate the extension of this formula to the flux F_n normal to an arbitrary surface in a solid at a point P, let coordinate axes be chosen with origin at P so that the xy-plane is tangent to the isotherm through P (Fig. 3). Let λ, μ, ν be the direction cosines of the normal n of the given surface. Now let the surface be displaced parallel to itself through a distance p, so that its tangent plane and the coordinate planes bound an elementary volume in the form of a tetrahedron.

If ΔA is the area of the face QRS made by the tangent plane, $\nu\Delta A$ is the area of the face in the xy-plane. As p approaches zero, the rate of flow of heat into the element through one of these faces must approach the rate of flow out through the other:

$$F_n\Delta A = F_z\nu\Delta A,$$

where F_z is the flux through the isotherm. The remaining two faces are perpendicular to the isotherm, so that the flux of heat through them is zero.

According to formula (1), $F_z = -K\, \partial u/\partial z$, so that

$$F_n = -K\nu \frac{\partial u}{\partial z}.$$

But according to the formula for the directional derivative,

$$\partial u \qquad \lambda \frac{\partial u}{\quad} + \mu \frac{\partial u}{\quad} + \qquad \partial u \qquad \partial u$$

since $\partial u/\partial x$ and $\partial u/\partial y$ are both zero, owing to the fact that x and y are distances along the isothermal surface. It follows that

$$F_n = -K \frac{\partial u}{\partial n}; \tag{2}$$

that is, *the flux of heat through any surface in the direction of the normal to that surface is proportional to the rate of change of the temperature with respect to distance along that normal.*

In the derivation of relation (2) it was assumed that there is no source of heat in the neighborhood of the point P, and that the derivatives of the temperature function u exist. Furthermore, our argument involved approximations, such as the use of tangent planes in place of surfaces, the validity of which use was not examined.

We shall not attempt to make the derivation of relation (2) precise. In a rigorous development of the mathematical theory of heat conduction, relation (2) can be postulated instead of (1). The results which follow from (2)—in particular, the heat equation derived in the next section—have long been known to agree with experimental measurements.

It should be observed that the temperature u serves as the potential function from which the flux of heat is obtained by finding its directional derivative. In the case of the gravitational or electrical potential, the directional derivative gives, respectively, the gravitational force or the flux of electricity in that direction; the flux of electricity is the current per unit area of surface normal to the direction.

9. The Heat Equation. Let $u(x, y, z, t)$ represent the temperature at a point P (x, y, z) of a solid at time t, and let K be the thermal conductivity of this solid, where K may be a function of x, y, z and t, or of u. Suppose that the point P is enclosed

by any surface S lying entirely within the solid, and let n represent the outward-drawn normal to the closed surface S. Then according to the formula for the flux in Sec. 8, the time rate of flow of heat into the volume V enclosed by S, through the surface S, is

$$\iint_S K \frac{\partial u}{\partial n}\, dS. \tag{1}$$

Now if δ is the density of the solid and c its specific heat, or the amount of heat required to raise the temperature of a unit mass of the solid 1 degree, another expression for the rate of increase of heat in the volume V is

$$\iiint_V c\delta \frac{\partial u}{\partial t}\, dV. \tag{2}$$

If λ, μ, ν are the direction cosines of the normal n, the integral (1) can be written

$$\iint_S \left(\lambda K \frac{\partial u}{\partial x} + \mu K \frac{\partial u}{\partial y} + \nu K \frac{\partial u}{\partial z}\right) dS.$$

This can be transformed, according to Green's theorem, into the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial u}{\partial z}\right)\right] dV,$$

which must be equal to the integral (2), so that

$$\iiint_V \left[\frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial u}{\partial z}\right) - c\delta \frac{\partial u}{\partial t}\right] dV = 0. \tag{3}$$

We are assuming that all the terms in the brackets in equation (3) are continuous functions in a neighborhood of P. Since the integral in equation (3) vanishes for every volume V, its integrand must vanish at P. For if the integrand were positive at P, its continuity would require that a sufficiently small volume V exists which contains P and throughout which the integrand is positive. The integral over V would then be positive, in

contradiction to equation (3). Similarly if the integrand were negative. Therefore, at P,

This is a general form of the *equation of conduction* of heat, or the *heat equation.*

It should be noted that we have assumed in the derivation that no sources of heat exist in the neighborhood of the point (x, y, z).

10. Other Cases of the Heat Equation. If the conductivity K is constant, or does not depend upon the coordinates, the heat equation becomes

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

where the coefficient k, called the *diffusivity,* is defined thus:

$$k = \frac{K}{c\delta}.$$

The equation appears most frequently in the form (1), or the abbreviated form

$$\frac{\partial u}{\partial t} = k\nabla^2 u. \tag{2}$$

The right-hand member can be expressed in terms of other coordinates by using the results of Sec. 7.

The heat equation is also called the *equation of diffusion.* It is satisfied by the concentration u of any substance which penetrates a porous solid by diffusion.

It was shown above that the temperature u everywhere within a solid satisfies the heat equation. To determine u as a definite function of x, y, z, and t, it is of course necessary to use, in addition to the heat equation, boundary conditions which describe the thermal state of the surface of the solid and the initial temperature. All these conditions make up the boundary value problem in the conduction of heat.

There are several special cases and simple generalizations of the heat equation which are important. First there are the cases in which the temperature is independent of one or more of the four independent variables, which consist of the space coordinates

and time t. If the temperatures are "steady"—that is, if u does not change with time—u satisfies Laplace's equation. This is approximately the case, for example, if the temperature distribution on the surface of a solid has been kept the same for a long period of time.

If conditions are such that there can be no flow of heat in the direction of the z-axis, the heat equation for "two-dimensional flow" applies:

$$\frac{\partial u}{\partial t} = k \qquad \cdot \frac{\partial^2 u}{\partial y^2}\Bigg).$$

Similarly for one-dimensional flow.

Continuous sources of heat may exist within a solid. If at each point (x, y, z) they supply heat at the rate of $F(x, y, z, t)$ units per unit time per unit volume, the heat equation becomes nonhomogeneous. For the case of one-dimensional flow, where the strength F of the source is a function of x and t, the equation becomes

$$\frac{\partial u}{\partial t} \quad \frac{\partial}{\partial x} \quad \cdot \frac{\partial u}{\partial x} \quad \cdot F(x, t). \tag{3}$$

This follows readily from the derivation in Sec. 9. Equation (3) may apply, for instance, to the temperature u in a wire which carries an electric current.

PROBLEMS

1. The lateral surface of a homogeneous prism is insulated against the flow of heat. The initial temperature is zero throughout, and the end $x = 0$ is kept at temperature zero while the end $x = L$ is kept at T_0, a constant temperature. Write the heat equation for this case.

Ans. $\partial u/\partial t = k(\partial^2 u/\partial x^2)$.

2. Find the steady temperature in Prob. 1, after the conditions given there have been maintained for a very long time. What is the flux through one end during the steady state?

Ans. $u = (T_0/L)x$; flux $= KT_0/L$.

3. State a physical problem whose solution is represented by the finite series in Prob. 5, Sec. 2.

4. Show that the temperature u in a uniform circular disk whose entire surface is insulated, and whose initial temperature is a function only of the distance r from the axis of the disk, satisfies the equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right).$$

5. If the initial temperature of a homogeneous sphere is a function only of the distance r from the center, and the surface is insulated, show that the temperature u of points inside satisfies the equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right).$$

11. The Equation of the Vibrating String. The transverse displacements of the points of a stretched string satisfy an important partial differential equation. Let the string be stretched between two fixed points on the x-axis and then given a displacement or velocity parallel to the y-axis. Its subsequent motion, with no external forces acting on it, is to be considered; this is described by finding the displacement y as a function of x and t.

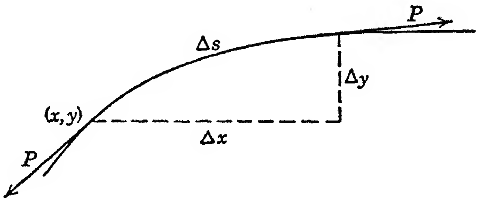

FIG. 4.

It will be assumed that δ, the mass per unit length, is uniform over the entire length of the string, and that the string is perfectly flexible, so that it can transmit tension but not bending or shearing forces. It will also be assumed that the displacements are small enough so that the square of the inclination $\partial y/\partial x$ can be neglected in comparison to 1; hence, if s is distance measured along the string at any instant,

$$\frac{\partial s}{\partial x} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} = 1,$$

approximately. The length of each part of the string therefore remains essentially unaltered, and hence the tension is approximately constant.

Consider the vertical components of the forces exerted by the string upon any element Δs of its length, lying between x and $x + \Delta x$ (Fig. 4). The y-component of the tensile force P exerted upon the element at the end (x, y) is

$$-P\frac{\partial y}{\partial s} = -P\frac{\partial y}{\partial x}\frac{\partial x}{\partial s} = -P\frac{\partial y}{\partial x},$$

approximately. The corresponding force at the end whose abscissa is $x + \Delta x$ is

where R is the usual factor in the remainder in Taylor's formula.

Setting the sum of these forces equal to the product of the mass of the element and the acceleration in the y-direction, we have

By dividing by Δx and letting Δx approach zero, it follows that

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

where

$$a^2 = \frac{P}{\delta}.$$

This is the *equation of the vibrating string;* it is also called the simple wave equation, since it is a special case of the wave equation of theoretical physics.

If an external force parallel to the y-axis acts along the string, it is easily seen that the equation becomes

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + F(x, t), \tag{2}$$

where $\delta F(x, t)$ is the force per unit length of string. In case the weight of the string is to be considered, for instance, the function F becomes the constant g, the acceleration of gravity.

If the transverse displacements are not confined to the xy-plane, two equations of type (2) are found, one involving the y, the other the z, of the points of the string, while the acceleration F is replaced by the y and z components of the external acceleration in those two equations, respectively.

Equation (1) is also satisfied by the longitudinal displacements in a homogeneous elastic bar; y is then the displacement along the bar of any point from its position of equilibrium. A column of air may be substituted for the bar, and the equation becomes one of importance in the theory of sound. The equation also applies to the torsional displacements in a right circular cylinder.

PROBLEMS

1. Derive equation (2) above.

2. State Prob. 5, Sec. 4, as a problem of displacements in a stretched string of infinite length. Show that the motion given by the result of that problem can be described as the sum of two displacements, obtained by separating the initial displacement into two equal parts, one of which moves to the left along the string with the velocity a, and the other to the right with the same velocity.

3. If a damping force proportional to the velocity, such as air resistance, acts upon the string, show that the equation of motion has the form

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - b \frac{\partial y}{\partial t},$$

where b is a positive constant.

12. Other Equations. Types. Some further partial differential equations of importance in the applications will be described briefly at this point. For their derivation and complete description, the reader should refer to books on the subjects involved.

A natural generalization of equation (1) of the last section is the *equation of the vibrating membrane:*

$$(1) \qquad \frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right).$$

Here the position of equilibrium of the stretched membrane is the xy-plane, so that z is the transverse displacement of any point from that position. Assumptions similar to those in the case of the string are necessary. The membrane is assumed to be thin and perfectly flexible, with uniform mass δ per unit area. The tensile stress P, or tension per unit length across any line, is assumed to be large, and the displacements small. The constant a^2 is then the ratio P/δ.

The telegraph equation,

$$(2) \qquad \frac{\partial^2 v}{\partial x^2} = KL \frac{\partial^2 v}{\partial t^2} + (RK + SL) \frac{\partial v}{\partial t} + RSv,$$

is satisfied by either the electric potential or the current in a long slender wire with resistance R, the electrostatic capacity K, the leakage conductance S, and the self-inductance L, all per

unit length of wire. The simple wave equation is a special case of this.

The *transverse displacements* $y(x, t)$ *of a uniform beam* satisfy the fourth-order equation

where the constant c^2 depends upon the stiffness and mass of the beam.*

Airy's *stress function* $\varphi(x, y)$, used in the theory of elasticity, satisfies the fourth-order equation

$$\frac{}{\partial x^4} + 2\frac{}{\partial x^2\,\partial y^2} + \frac{}{\partial y^4} = 0, \tag{4}$$

often written $\nabla^4\varphi = 0$. It serves in a sense as a potential function from which shearing and normal stresses within an elastic body can be derived. The form (4) assumes that no deformations exist in the z-direction.

Linear partial differential equations of the second order with two independent variables x, y are classified into three types in the theory of these equations. If the terms of second order, when collected on one side of the equation, are

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2},$$

where A, B, C are constants, the equation is of *elliptic*, *parabolic*, or *hyperbolic* type according as $(B^2 - 4AC)$ is negative, zero, or positive. In the study of boundary value problems it will be observed that these three types require different kinds of boundary conditions to completely determine a solution.

Note that Laplace's equation in x and y is elliptic, while the heat equation and the simple wave equation in x and t are parabolic and hyperbolic, respectively. The telegraph equation is also hyperbolic, if $KL \neq 0$.

13. A Problem in Vibrations of a String. When the differential equation is linear and the boundary conditions consist of linear equations, the *boundary value problem* itself is called *linear*.

* See, for instance, Timoshenko, "Vibration Problems in Engineering," p. 221.

A method which can be used to solve a large class of such problems will now be illustrated. It will be seen that the process leads naturally to a problem in Fourier series. A formal solution of the following problem will be given.

Find the transverse displacements $y(x, t)$ in a string of length L stretched between the points $(0, 0)$ and $(L, 0)$ if it is displaced initially into a position $y = f(x)$ and released from rest at this position with no external forces acting.

The required function y is the solution of the following boundary value problem:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (t > 0,\ 0 < x < L), \tag{1}$$

$$y(0, t) = 0, \qquad y(L, t) = 0 \qquad (t \geqq 0), \tag{2}$$

$$y(x, 0) = f(x) \qquad (0 \leqq x \leqq L), \tag{3}$$

$$\frac{\partial y(x, 0)}{\partial t} = 0 \qquad (0 \leqq x \leqq L). \tag{4}$$

Our method consists of finding particular solutions of the partial differential equation (1) which satisfy the homogeneous boundary conditions (2) and (4), and then of determining a linear combination of those solutions which satisfies the nonhomogeneous boundary condition (3).

Particular solutions of equation (1) of the type

$$y = XT, \tag{5}$$

where X is a function of x alone and T a function of t alone, can easily be found by means of ordinary differential equations.

According to equation (5), $\partial y/\partial x = X'T$, $\partial y/\partial t = XT'$, etc., where the prime denotes the ordinary derivative with respect to the only independent variable involved in the function. Substituting into equation (1), we find

$$XT'' = a^2 X''T,$$

or, upon separating the variables by dividing by a^2XT,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{}.$$

Since the member on the left is a function of x alone, it cannot vary with t; it is equal to a function of t alone, however, and

thus it cannot vary with x. Hence both members must be equal to a constant, say γ, so that

$$X''(x) - \gamma X(x) = 0, \tag{6}$$

$$T''(t) - \gamma a^2 T(t) = 0. \tag{7}$$

If our particular solution is to satisfy conditions (2), XT must vanish when $x = 0$ and when $x = L$, for all values of t involved. Therefore

$$X(0) = 0, \qquad X(L) = 0. \tag{8}$$

Similarly, if it is to satisfy condition (4),

$$T'(0) = 0. \tag{9}$$

Equations (6) and (7) are linear homogeneous ordinary differential equations with constant coefficients. The auxiliary equation corresponding to (6), $m^2 - \gamma = 0$, has the roots $m = \pm\sqrt{\gamma}$. The general solution of equation (6) is therefore

$$X = C_1 e^{x\sqrt{\gamma}} + C_2 e^{-x\sqrt{\gamma}},$$

where C_1 and C_2 are arbitrary constants. But if γ is positive, it is easily seen that there are no values of C_1 and C_2 for which this function X satisfies both of conditions (8).

Suppose γ is negative, and write

$$\gamma = -\beta^2.$$

The general solution of equation (6) can then be written

$$X = A \sin \beta x + B \cos \beta x,$$

where A and B are arbitrary constants.

If $X(0) = 0$, the constant B must vanish. Then A must be different from zero, since we are not interested in the trivial solution $X(x) \equiv 0$. So if $X(L) = 0$, we must have

$$\sin \beta L = 0.$$

Hence there is a discrete set of values of β, namely,

$$\beta = \frac{n\pi}{L} \qquad (n = 1, 2, \cdots),$$

for which the system consisting of equation (6) and conditions (8) has solutions. These solutions are

$$X = A \sin \frac{n\pi x}{L}.$$

Note that no new solutions are obtained when $n = -1, -2, -3, \cdots$.

Substituting $-n^2\pi^2/L^2$ for γ in differential equation (7) and applying condition (9), we find that

$$T = C \cos \frac{n\pi at}{L},$$

where C is an arbitrary constant.

Therefore all the functions

$$A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} \qquad (n = 1, 2, \cdots) \tag{10}$$

are solutions of our partial differential equation (1) and satisfy the linear homogeneous conditions (2) and (4), when $A_1, A_2, \cdots$ are arbitrary constants.

Any finite linear combination of these solutions will also satisfy the same conditions (Theorem 1, Chap. I); but when $t = 0$, it will reduce to a finite linear combination of the functions $\sin (n\pi x/L)$. Thus condition (3) will not be satisfied unless the given function $f(x)$ has this particular character.

Consider an infinite series of functions (10),

$$y = \sum_1^\infty A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}. \tag{11}$$

This satisfies equation (1) provided it converges and is termwise differentiable (Theorem 2, Chap. I); it also satisfies conditions (2) and (4). It will satisfy the nonhomogeneous condition (3) provided the numbers A_n can be so determined that

$$f(x) = \sum_1^\infty A_n \sin \frac{n\pi x}{L}. \tag{12}$$

It will be shown in Sec. 15 that if such an expansion of $f(x)$ is possible, the numbers A_n must have the values

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{13}$$

Equation (11) with coefficients (13) is formally the solution of the boundary value problem (1)-(4).

The series on the right of equation (12) with the coefficients defined by (13) is called the *Fourier sine series* of the function $f(x)$. In a later chapter it will be shown that this series actually converges to the function $f(x)$ in the interval $0 \leqq x \leqq L$, provided $f(x)$ satisfies certain moderate conditions—conditions which are almost always satisfied by functions which arise in the applications.

Other questions are left unsettled at this point in the treatment of this problem. Series (11) has not been shown to be convergent, or to represent a continuous function, or to be termwise differentiable twice with respect to either x or t. It has not

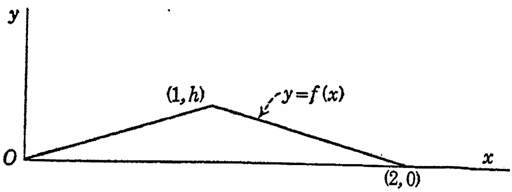

FIG. 5.

been shown that series (11) is the only solution of the problem (1)-(4). Questions of this character are to be treated later on.

14. Example. The Plucked String. As a special case of the problem just treated, let the string be stretched between the points (0, 0) and (2, 0), and suppose its mid-point is raised to a height h above the x-axis. The string is then released from rest in this broken-line position (Fig. 5).

The function $f(x)$ which describes the initial position can be written, in this case,

$$\begin{aligned} f(x) &= hx && \text{when } 0 \leqq x \leqq 1, \\ &= -hx + 2h && \text{when } 1 \leqq x \leqq 2. \end{aligned}$$

The coefficients in solution (11), Sec. 13, are, according to formula (13), Sec. 13,

$$\begin{aligned} A_n &= \int_0^2 f(x) \sin \qquad dx \\ &= h \int_0^1 x \sin \frac{n\pi x}{2} \qquad\qquad \cdot 2) \sin \frac{n\pi x}{2}\, dx. \end{aligned}$$

After integrating and simplifying, we find that

$$A \qquad 8h \quad . \quad n\pi$$

so that the displacement $y(x, t)$ in this case of the plucked string is given by the formula

$$\frac{8h}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}\cos\frac{n\pi at}{2}$$

$$8h \qquad \frac{\pi at}{2} - \frac{1}{9}\sin\frac{3\pi x}{2}\cos\frac{3\pi at}{2}$$

$$+ \frac{1}{}\sin\frac{5\pi x}{}\cos 5\pi at$$

Another form of this solution will be obtained later [formula (4), Sec. 43].

15. The Fourier Sine Series. In the solution of the problem of Sec. 13 it was necessary to determine the coefficients A_n so that the series of sines would converge to $f(x)$. Assuming that an expansion of the type needed there, namely,

(1) $$= A_1 \sin \pi x + A_2 \sin \frac{}{L} + \qquad + A_n \sin\frac{n\pi x}{L} +$$

is possible when $0 \leqq x \leqq L$, and that the series can be integrated term by term after being multiplied by $\sin (n\pi x/L)$, it is easy to see what values the coefficients must have.

It is necessary to recall that

$$\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} \quad \frac{1}{2} \qquad \frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\Bigg],$$

$$\sin^2 m\pi x$$

and hence, when m and n are integers,

(2) $$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0 \qquad \text{if}$$

$$\frac{L}{2} \qquad \text{if } m = n.$$

The functions $\sin (n\pi x/L)$ $(n = 1, 2, \cdots)$ therefore form an *orthogonal system* in the interval $0 < x < L$; that is, the integral

over that interval of the product of any two distinct functions of the system is zero.

Now let all terms in equation (1) be multiplied by sin $(n\pi x/L)$ and integrated between 0 and L. The first term on the right becomes

$$A_1 \int_0^L \sin \frac{\pi x}{L} \sin \frac{n\pi x}{L}\, dx.$$

This is zero unless $n = 1$, according to the orthogonality property (2). Likewise all terms on the right except the nth one become zero; so the process gives

$$\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = A_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \frac{A_n L}{}$$

according to property (2). Hence the coefficients in equation (1) must have the values

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{3}$$

The Fourier sine series corresponding to $f(x)$ can be written

$$f(x) \sim \frac{2}{L} \sum_1^\infty \sin \frac{n\pi x}{L} \int_0^L f(\xi) \sin \frac{n\pi \xi}{L}\, d\xi, \tag{4}$$

where the sign $\sim$ is used here to denote correspondence. It is to be shown later on that the series does converge to $f(x)$ in general.

PROBLEMS

1. Show that the Fourier sine series corresponding to $f(x) = 1$ in the interval $0 < x < \pi$ is

$$1 \sim \frac{4}{\pi}\left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots\right).$$

2. Show that the sine series for $f(x) = x$ in the interval $0 < x < 1$ is

$$x \sim \frac{\quad}{\quad} \sum_1 \frac{\quad}{\quad} \sin$$

3. Find the solution of the problem of the string in Sec. 13 if the initial displacement is $f(x) = A \sin (\pi x/L)$. Discuss the motion.

Ans. $y = A \sin (\pi x/L) \cos (\pi a$

16. Imaginary Exponential Functions. According to the power series expansion of e^z,

$$e^{ix} = \qquad n!$$

$$= 1 - \frac{x^{-}}{2!} + \frac{x}{4!} -$$

where $i = \sqrt{-1}$. So

(1) $$= \cos x + i \sin x.$$

This is usually taken as the definition of the exponential function with imaginary exponents. Then

$$e^{-ix} =$$

and by first eliminating cos x and then sin x between this equation and equation (1), we find that

(2) $$\sin x = \frac{e^{ix} - e^{-ix}}{2} = \sinh (ix),$$

(3) $$\cos x = \quad + \quad = \cosh (ix).$$

When the coefficients of a linear homogeneous differential equation are constant, particular solutions in the form of exponential functions can be found.

To illustrate the use of exponential functions in partial differential equations, consider again the problem in Sec. 13. The function

$$y = e^{\alpha x + \beta t},$$

where α and β are constants, is clearly a solution of the equation

provided that

Hence the functions

$$e^{\alpha x} e^{\pm a \alpha t}, \quad e^{-\alpha x} e^{\pm a \alpha t}$$

are solutions.

Except for a constant factor, the difference between the two products just written is the only linear combination which

vanishes at $x = 0$ [condition (2), Sec. 13]. Thus the functions

$$e^{\pm \alpha a t} \sinh \alpha x$$

satisfy that condition as well as equation (4). The linear combination of these two functions which satisfies the condition that $\partial y/\partial t = 0$ when $t = 0$ is the sum, or

$$\sinh \alpha x \cosh \alpha a t. \tag{5}$$

But here α must be imaginary, since our function is to vanish when $x = L$, because the hyperbolic sine of a real argument vanishes only when the argument is zero. According to equations (2) and (3), when $\alpha = i\mu$ the function (5) can be written, except for a constant factor, as

$$\sin \mu x \cos \mu a t.$$

This vanishes when $x = L$ if $\mu = n\pi/L$.

Thus we again have the particular solutions

$$A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}$$

of equation (4), which satisfy the homogeneous conditions in the problem in Sec. 13. From this point on the procedure is the same as in that section.

As another application of imaginary exponential functions, note that

$$2(\cos \theta + \cos 2\theta + \cdot \cdot \cdot + \cos N\theta) = \sum_1^N e^{in\theta} + \sum_1^N e^{-in\theta}.$$

Summing the finite geometric series on the right, this becomes

$$2 \sum^N \cos n\theta = \frac{e^{i\theta}(1 - e^{iN\theta})}{1 - e^{i\theta}} + \frac{e^{-i\theta}(1 - e^{-iN\theta})}{1 - e^{-i\theta}}$$

$$= \frac{\qquad}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}$$

This can be written at once in the form

$$\frac{(1}{\sin \cdot} \tag{6}$$

which is known as *Lagrange's trigonometric identity.* This identity will be useful in the theory of Fourier series.

PROBLEMS

1. Use exponential functions to determine particular solutions of the simple heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

which vanish when $x = 0$ and $x = \pi$. (Compare Prob. 5, Sec. 2.)

2. Use exponential functions to determine particular solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

such that $u = 0$ when $y = 0$, and $\partial u/\partial x = 0$ when $x = 0$ and $x = 1$.

Ans. $u = A_n \cos n\pi x \sinh n\pi y \qquad (n = 0, 1, 2, \cdots)$.

CHAPTER III

ORTHOGONAL SETS OF FUNCTIONS

17. Inner Product of Two Vectors. Orthogonality. The concept of an orthogonal set of functions is a natural generalization of that of an orthogonal set of vectors, that is, a set of mutually perpendicular vectors. In fact, a function can be considered as a generalized vector, so that the fundamental properties of the set of functions are suggested by the analogous properties of the set of vectors. In the following discussion of simple vectors, the terminology and notation which apply to the generalized case will be used whenever it seems advantageous for the later generalizations.

Let either g or $g(r)$ denote a vector in ordinary three-dimensional space whose rectangular components are the three numbers $g(1)$, $g(2)$, and $g(3)$. It is the radius vector of the point having these numbers as rectangular cartesian coordinates. The square of the length of this vector, called its *norm*, will be written $N(g)$; it is the sum of the squares of the components of g:

$$N(g) = \qquad {}^2(2) \qquad = \sum g^2(r).$$

If $N(g) = 1$, g is a unit vector, also called a *normed* or a *normalized* vector.

Let θ be the angle between two vectors $g_1(r)$ and $g_2(r)$. Since the components $g_1(1)$, $g_1(2)$, $g_1(3)$ are proportional to the direction cosines of the vector g_1, and similarly for g_2, the formula from analytic geometry for $\cos\theta$ can be written

$$\frac{g_1(2)g_2(2) + g_1(3)g_2(3)}{}$$

The numerator on the right is called the *inner product* (or scalar product) of the vectors g_1 and g_2, denoted by the symbol (g_1, g_2); thus

(2)

$$= \sqrt{N(g_1)}\,\sqrt{N(g_2)}\cos\theta.$$

When $N(g_2) = 1$, $g_2)$ is the projection of the vector g_1 in the direction of g_2.

The condition that the vectors g_1 and g be orthogonal, or perpendicular to each other, can be written

(3) $$(g_1, g_2) = 0$$

or, in terms of components,

(4) $$g_1(r)g_2(r) = 0.$$

Note also that expression (1) for the norm of g can be written

$$N(g) = (g, g).$$

18. Orthonormal Sets of Vectors. Given an orthogonal set of three vectors g_n $(n = 1, 2, 3)$, a set of unit vectors φ_n having the same directions can be formed by dividing each component of g_n by the length of g_n. The components of φ_1, for instance, are $\varphi_1(r) = g_1(r)[N(g_1)]^{-\frac{1}{2}}$ $(r = 1, 2, 3)$. This set of mutually perpendicular unit vectors φ_n, obtained by normalizing the mutually perpendicular vectors g_n, is called an *orthonormal set.* Such a set can be described by means of inner products by writing

$$(\varphi_m, \varphi_n) = \qquad (m, n = 1, 2, 3),$$

where δ_{mn}, called Kronecker's δ, is 0 or 1 according as m and n are different or equal:

$$\begin{aligned} \delta_{mn} &= 0 \qquad \text{if } m \neq n, \\ &= 1 \qquad \text{if } m = n. \end{aligned}$$

The condition (1) therefore requires that each vector of the set φ_1, φ_2, φ_3 is perpendicular to every other one in that set, and that each has unit length.

The symbol $\{\varphi_n\}$ will be used to denote an orthonormal set whose vectors are φ_1, φ_2, and φ_3. The simplest example of such a set is that consisting of the unit vectors along the three coordinate axes.

Every vector f in the space considered can be expressed as a linear combination of the vectors φ_1, φ_2, and φ_3. That is, three numbers c_1, c_2, c_3 can be found for which

$$f(r) = c_1\varphi_1(r) + \qquad + \qquad (r = 1, 2, 3), \tag{2}$$

when the components $f(1)$, $f(2)$, $f(3)$ are given. To find the number c_1 in a simple way, consider equation (2) as a vector equation and take the inner product of both its members by φ_1. This gives

$$(f, \varphi_1) = c_1(\varphi_1, \varphi_1) + c_2(\varphi_2, \varphi_1) + c_3(\varphi_3, \varphi_1) = c_1,$$

since $(\varphi_1, \varphi_1) = 1$ and $(\varphi_2, \varphi_1) = (\varphi_3, \varphi_1) = 0$, according to condition (1). Similarly c_2 and c_3 are found by taking the inner product of the members of equation (2) by φ_2 and φ_3, respectively. The coefficients are therefore

$$c_n = (f, \varphi_n) = \sum_1^3 f(r)\varphi_n(r) \qquad (n = 1, 2, 3). \tag{3}$$

The representation (2) can then be written

$$\begin{aligned} f(r) &= (f, \varphi_1)\varphi_1(r) + (f, \varphi_2)\varphi_2(r) + (f, \varphi_3)\varphi_3(r) \\ &= \sum_1^3 (f, \varphi_n)\varphi_n(r). \end{aligned} \tag{4}$$

The representation (2) or (4) may be called an "expansion" of the arbitrary vector f in a finite series of the orthonormal reference vectors φ_1, φ_2, and φ_3. These orthogonal reference vectors were assumed to be normalized only as a matter of convenience, in order to obtain the simple formulas (3) for the coefficients in the expansion. The normalization is not necessary, of course.

The definitions and results just given can be extended immediately to vectors in a space of k dimensions. In this case the index r, which indicates the component, has values from 1 to k, instead of 1 to 3; similarly the indices m and n, which distinguish the different vectors of an orthonormal set, run from 1 to k. The definition of the inner product of the vectors g_1 and g_2 in this space, for instance, becomes

$$(g_1, g_2) = \sum_{r=1}^k g_1(r)g_2(r). \tag{5}$$

The formal extension to vectors in a space of a countably infinite number of dimensions ($k = \infty$) is also possible. In this case the numbers $g(r)$ ($r = 1, 2, \cdots$) which define a vector g would be so restricted that the infinite series involved, such as the series in (5) with k infinite, would converge. The possibility of the representation corresponding to (2) would have to be examined, of course.

A generalization of another sort is also possible. The units of length on the rectangular coordinate axes, with respect to which the components of vectors are measured, may vary from one axis to another. In such a case the scalar product of two vectors g_1 and g_2 in three-dimensional space has the form

$$r=1$$

The "weight numbers" $p(1)$, $p(2)$, and $p(3)$ here depend upon the units of length used along the three axes.

19. Functions as Vectors. Orthogonality. A vector $g(r)$ in three dimensions was described above by the numbers $g(1)$, $g(2)$, $g(3)$. Any function $g(r)$ which has real values when $r = 1, 2, 3$ will represent a vector if it is agreed that these values are the components of the vector. This function may not be defined for any other values of r, in which case its graph would consist only of three points.

The function $g(r)$ will represent a vector in space of k dimensions if it has real values when $r = 1, 2, \cdots, k$, which are considered as the components of the vector. If $g(r)$ is defined only at these points, it is determined by the vector; graphically it is represented by k points whose abscissas are $r = 1, 2, \cdots, k$, and whose ordinates are the corresponding components of the vector.

Now let $g(x)$ be a function defined for all values of x in an interval $a \leqq x \leqq b$. To consider this function as a vector, the components should consist of all the ordinates of its graph in the interval. The argument x, which has replaced r here, has as many values as there are points in the interval, so that the number of components is not only infinite but uncountable. It is therefore impossible to sum with respect to x as we do with the index r. The natural process now is to sum by integration.

The *norm* of the function or vector $g(x)$, or the sum of the squares of its components, is therefore defined as the number

$$N(g) = \int_a^b [g(x)]^2\,dx. \tag{1}$$

The *inner product* of two functions $g_m(x)$ and $g_n(x)$ is defined as the number

$$(g_m, g_n) = \int_a^b g_m(x)g_n(x)\,dx, \tag{2}$$

in analogy to equation (5), Sec. 18. The condition that the two functions be *orthogonal* is written

$$(g_m, g_n) = 0,$$

or

$$\int_a^b g_m(x)g_n(x)\,dx = 0. \tag{3}$$

Just as before, definition (1) can be written $N(g) = (g, g)$.

A set (or system) of functions $\{g_n(x)\}$ $(n = 1, 2, \cdots)$ is orthogonal in the interval (a, b) if condition (3) is true when $m \neq n$ for all functions of the set. The functions of the set are normed by dividing each function $g_n(x)$ by $[N(g_n)]^{\frac{1}{2}}$, thus forming a set $\{\varphi_n(x)\}$ $(n = 1, 2, \cdots)$, which is normed and orthogonal, or *orthonormal.* An orthonormal system in (a, b) is then characterized as follows:

$$(\varphi_m, \varphi_n) = \delta_{mn} \qquad (m, n = 1, 2, \cdots), \tag{4}$$

where δ_{mn} is Kronecker's δ, defined in Sec. 18. Written in full, equation (4) becomes

$$\begin{aligned} \int_a^b \varphi_m(x)\varphi_n(x)\,dx &= 0 \quad \text{if } m \neq n, \\ &= 1 \quad \text{if } m = n, \ (m, n = 1, 2, \cdots). \end{aligned} \tag{5}$$

The interval (a, b) over which the functions and their inner products are defined is called the *fundamental interval.* Functions for which the integrals representing the inner product and the norm fail to exist must, of course, be excluded.

Throughout this book, only *functions which are bounded and integrable in the fundamental interval, and whose norms are not zero,* will be considered. The aggregate of all such functions for the given interval makes up the *function space* being considered,

in just the same way that the three-dimensional vector space consists of all vectors with three components $g(r)$ $(r = 1, 2, 3)$.

An example of an orthogonal set of functions has already been given in Sec. 15; namely, the functions

$$\sin\frac{n\pi x}{L} \qquad (n = 1, 2, \cdots).$$

The fundamental interval is the interval $(0, L)$. The norm of all these functions is the same number, $L/2$, so the orthonormal set consists of the functions

$$\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad (n = 1, 2, \cdots).$$

The set $\{\sin(n\pi x/L)\}$ is also orthogonal in the interval $(-L, L)$; the normalizing factor is easily seen to be $1/\sqrt{L}$ in this case.

PROBLEMS

1. Show that the set of functions $\{\cos nx\}$ $(n = 0, 1, 2, \cdots)$ is orthogonal in the interval $(0, \pi)$. What is the corresponding orthonormal set? *Ans.* $\{1/\sqrt{\pi}, \sqrt{2/\pi}\cos nx\}$ $(n = 1, 2, \cdots)$.

2. Show that the set $\{\sin x, \sin 2x, \sin 3x, \cdots, 1, \cos x, \cos 2x, \cdots\}$ is orthogonal in the interval $(-\pi, \pi)$. Normalize this set.

20. Generalized Fourier Series. Given a countably infinite orthonormal set of functions $\{\varphi_n(x)\}$ $(n = 1, 2, \cdots)$, it may be possible to represent an arbitrary function in the fundamental interval as a linear combination of the functions $\varphi_n(x)$,

$$(1) \quad f(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x) + \cdots \qquad (a < x < b).$$

This corresponds to representation (2), Sec. 18, of any vector in terms of the vectors of an orthonormal set.

If the series in equation (1) converges and if, after being multiplied by $\varphi_n(x)$, it can be integrated term by term over the fundamental interval (a, b), the coefficients c_n can be found in the same way as before. Writing the inner product of both members of equation (1) by $\varphi_n(x)$—that is, multiplying (1) by φ_n and integrating over (a, b)—we have

$$\begin{aligned}(f, \varphi_n) &= c_1(\varphi_1, \varphi_n) + c_2(\varphi_2, \varphi_n) + \cdots + c_n(\varphi_n, \varphi_n) + \cdots \\ &= c_n,\end{aligned}$$

since $(\varphi_m, \varphi_n) = \delta_{mn}$. That is, c_n is the projection of the vector f on the unit vector φ_n.

These numbers c_n are called the *Fourier constants* of $f(x)$ corresponding to the orthonormal system $\{\varphi_n(x)\}$; they can be written

$$(2) \qquad c_n = \int_a^b f(x)\varphi_n(x)\,dx \qquad (n = 1, 2, \cdots).$$

The series in (1) with these coefficients is called the *generalized Fourier series corresponding to* $f(x)$, written

$$(3) \qquad f(x) \sim \sum_1^\infty c_n\varphi_n(x) = \sum_1^\infty \varphi_n(x$$

The above correspondence between $f(x)$ and its series will not always be an equality. This can be anticipated at once by considering the case of vectors in three dimensions. In that case if only two vectors $\varphi_1(r)$, $\varphi_2(r)$, make up the orthonormal system, any vector not in the plane of those two could not be represented in the form $c_1\varphi_1(r) + c_2\varphi_2(r)$. The reference system here is not complete, in the sense that there is a vector in the three-dimensional space which is perpendicular to both of its vectors φ_1 and φ_2.

Likewise in formula (3), if $f(x)$ is orthogonal to every member $\varphi_n(x)$ of the system, every term in the series on the right is zero, and so the series does not represent $f(x)$.

If there is no function in the space considered which is orthogonal to every $\varphi_n(x)$, the system $\{\varphi_n(x)\}$ is called complete. So the system must necessarily be complete if all functions are to be represented by their generalized Fourier series with respect to that system.

PROBLEMS

1. Show that the set $\{\sqrt{2/L}\cos(n\pi x/L)\}$ $(n = 1, 2, \cdots)$ is orthonormal in the interval $(0, L)$, but not complete without the addition of a function corresponding to $n = 0$.

2. Show that the system $\{\sin n\pi x\}$ $(n = 1, 2, \cdots)$ is orthogonal but not complete in the interval $(-1, 1)$.

21. Approximation in the Mean. Let $K_m(x)$ represent a finite linear combination of m functions of an orthonormal set $\{\varphi_n(x)\}$ $(n = 1, 2, \cdots; a \leqq x \leqq b)$; that is,

$$(1) \qquad K_m(x) = \gamma_1\varphi_1(x) + \gamma_2\varphi_2(x) + \cdots + \gamma_m\varphi_m(x).$$

The values of the constants γ_n can easily be found for which $K_m(x)$ is the best *approximation in the mean* to any given function $f(x)$; this means the best approximation in the sense that the value of the integral

$$J = \int_a^b [f(x) - K_m(x)]^2 \, dx \tag{2}$$

is to be as small as possible; it is also the approximation in the sense of *least squares*.

Writing c_n for the Fourier constants of $f(x)$ with respect to $\varphi_n(x)$,

$$c_n = \int_a^b f(x)\varphi_n(x) \, dx,$$

the integral J can be written

$$J = \int_a^b [f(x) - \qquad \gamma_m\varphi_m(x)]^2 \, dx$$
$$= \int_a^b [f(x)]^2 \, dx$$

Completing the squares here by adding and subtracting c_1^2, $c_2^2, \cdots, c_m^2$, gives

$$J = \qquad (\gamma_1 - \qquad (\gamma_2 - \tag{3}$$

It is clear from (2) that $J \geqq 0$, so it follows from equation (3) that J has its least value when $\gamma_1 = c_1, \gamma_2 = c_2, \cdots, \gamma_m = c_m$. The result can be stated as follows:

Theorem 1. *The Fourier constants of a function $f(x)$ with respect to the functions $\varphi_1(x), \varphi_2(x), \cdots, \varphi_m(x)$ of an orthonormal set are those coefficients for which a linear combination $K_m(x)$ of these functions is the best approximation in the mean to $f(x)$, in the fundamental interval (a, b).*

Since $J \geqq 0$, it follows from equation (3), by taking $\gamma_n = c_n$, that

$$c_2^2 + \qquad \leqq \int_a^b [f(x)]^2 \, dx. \tag{4}$$

This is known as *Bessel's inequality*. The number on the right is independent of m; so it follows that the series of squares of the

Fourier constants of any function

$$c_1^2 + c_2^2 + \cdots + c_n^2 + \cdots = \sum_1^\infty c_n^2$$

always converges; and its sum is not greater than the norm of $f(x)$

$$(5) \qquad c_n^2 \leqq \int_a^b [f(x)]^2\, dx.$$

It follows that *the Fourier constants of every function corresponding to any orthonormal system* $\{\varphi_n\}$ *approach zero as n tends to infinity:*

$$(6) \qquad \lim_{n\to\infty} c_n = 0;$$

because a necessary condition for the convergence of the series in (5) is that its general term c_n^2 approaches zero as n becomes infinite.

22. Closed and Complete Systems. Let $S_m(x)$ be the sum of m terms of the generalized Fourier series corresponding to $f(x)$, with respect to an orthonormal set of functions $\{\varphi_n\}$ $(n = 1, 2, \cdots)$; that is,

$$(1) \qquad S_m(x) = \sum_1^m c_n\varphi_n(x).$$

This is the sum $K_m(x)$ in the last section when $\gamma_n = c_n$.

The sum $S_n(x)$ is said to *converge in the mean* to the function $f(x)$ if

$$(2) \qquad \lim_{m\to\infty} \int_a^b [f(x) - S_m(x)]^2\, dx = 0.$$

This is also written

$$\underset{m\to\infty}{\text{l.i.m.}}\ S_m(x) = f(x),$$

where the abbreviation l.i.m. stands for *limit in the mean.*

If the relation (2) is true for each $f(x)$ in the function space considered, the system $\{\varphi_n(x)\}$ is said to be *closed.** According to Theorem 1, then, the system is closed if every function can be

* The definitions of the terms closed and complete (Sec. 20) given here are those most commonly used today. Many German writers use the term closed (*abgeschlossen*) to denote what we have called complete, and complete (*vollständig*) for our concept of closed.

approximated arbitrarily closely in the mean by some linear combination of the functions $\varphi_n(x)$.

By expanding the integrand in equation (2) and keeping the definition of c_n in mind, we have

$$\lim_{m\to\infty} \left\{ \int_a^b \qquad]^2\, dx - 2 \qquad {}_1 \qquad {}_1 \right. \qquad = 0.$$

Hence *for every closed system it is true that*

$$\sum_1^\infty c_n^2 = \int_a^b [f(x)]^2\, dx. \tag{3}$$

This is known as *Parseval's theorem.* When written in the form

(4)

it identifies the sum of the squares of the components of f, with respect to the reference vectors φ_n, with the norm of f.

Suppose $\theta(x)$ is a function which is orthogonal to every function of the closed set. Substituting it for f in equation (4) gives $N(\theta) = 0$, so that $\theta(x)$ cannot belong to the function space; and thus it is shown that the set is complete (Sec. 20). The following theorem is therefore established:

Theorem 2. *If the set* $\{\varphi_n(x)\}$ *is closed, it is complete.*

It is an immediate consequence that if there is a function which is orthogonal to every member of the set, the set cannot be closed.

This is only a bare introduction to a general theory which has been developed extensively in recent years. To carry it further (even to prove the converse of Theorem 2), a broader class of functions and the idea of the Lebesgue integral are needed.

But the term "closed" was defined here with respect to convergence in the mean, and this type of convergence does not guarantee ordinary convergence at any point. That is, the statement (2) is quite different from the statement of ordinary convergence:

$$\lim S_m(x) = f(x) \qquad (a \leqq x \leqq b).$$

It is this ordinary convergence, and the concept of closed orthog-

onal sets with respect to it, which are usually needed in the applications.

No general tests of a practical nature exist for showing that a set of functions is closed. That is another reason for deserting the general theory at this point.

23. Other Types of Orthogonality. Some of the important extensions of the concept of orthogonal sets of functions should be noted.

a. A set $\{g_n(x)\}$ $(n = 1, 2, \cdots)$ is orthogonal in an interval (a, b) with respect to a given *weight function* $p(x)$, where it is usually supposed that $p(x) \geqq 0$ in (a, b), if

$$(1) \qquad \int_a^b p(x)g_m(x)g_n(x)\,dx = 0 \quad \text{when } m \neq n \ (m, n = 1, 2, \cdots).$$

The integral on the left represents the inner product (g_m, g_n) with respect to the weight function, a generalization of the inner product of vectors in terms of components with respect to axes along which different units of length are used (Sec. 18). The norm of $g_n(x)$ in this case is, of course,

$$N(g_n) = (g_n, g_n) = \int_a^b p(x)[g_n(x)]^2\,dx \quad (n = 1, 2, \cdots).$$

By multiplying each function g_n of the set by the normalizing factor $[N(g_n)]^{-\frac{1}{2}}$, the corresponding orthonormal set is obtained.

This type of orthogonality can be reduced at once to the ordinary type having the weight function 1. It is only necessary to use the products $\sqrt{p(x)}g_n(x)$ as the functions of the system; then equation (1) shows that the system so formed has ordinary orthogonality in the interval (a, b).

An important instance of orthogonality with respect to weight functions will be seen in the study of Bessel functions later on.

The *Tchebichef polynomials*,

$$(2) \qquad T_n(x) = \frac{1}{2^{n-1}} \cos\,(n \arccos x), \qquad T_0(x) = 1 \qquad (n = 1, 2, \cdots),$$

also form a set of this type. This set is orthogonal in the interval $(-1, 1)$ with respect to the weight function

$$p(x) = (1 - x^2)^{-\frac{1}{2}}.$$

This is easily verified by integration; thus,

$$\int_{-1}^{1} T_m(x)T_n(x) \quad dx \quad 1 \quad \cos m\theta \cos n\theta \, d\theta$$
$$= 0 \qquad \text{if } m \neq n.$$

b. Another extension of orthogonality applies to a system of complex functions of a real variable x $(a \leqq x \leqq b)$. A system consisting of the functions $g_n(x)$, where

$$g_n(x) = u_n(x) + iv_n(x),$$

is said to be orthogonal in the *Hermitian sense* if

$$\int_a^b g_m(x)\overline{g_n(x)}\, dx = 0 \qquad \text{when } m \neq n, \tag{3}$$

where $\overline{g_n(x)} = u_n(x) - iv_n(x)$, the conjugate of g_n. The system is normed if

$$\int_a^b g_n(x)\overline{g_n(x)}\, dx = 1;$$

that is, if

for every n.

When the functions are real, $v_n(x) = 0$, and this type reduces to the ordinary orthogonality.

Imaginary exponential functions furnish the most important examples of such systems. For instance, the functions

$$= \cos nx + i \sin nx \quad (n = 0, \pm 1, \pm 2, \tag{4}$$

form a system which is orthogonal on the interval $(-\pi, \pi)$ in the above sense. The proof is left as a problem.

c. Extensions to cases in which the fundamental interval is infinite in length are obtained by replacing a by $-\infty$ or b by ∞, or both.

d. For systems $\{g_n(x, y)\}$ $(n = 1, 2, \cdots)$ of functions of two variables, the fundamental interval is replaced by a region in the xy-plane, and the integrations are carried out over this region. Similar extensions apply when three or more variables are present. Weight functions may be introduced in such cases too, as well as in case *c*.

PROBLEMS

1. By using the binomial expansion and equating real parts in the well-known formula

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$$

obtain the identity

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta -$$

where $\binom{n}{k}$ are the binomial coefficients, and $A_n = i^n \sin^n \theta$ if n is even, $A_n = ni^{n-1} \cos \theta \sin^{n-1} \theta$ if n is odd. Hence show that the functions $T_n(x)$ defined by equations (2) above are actually polynomials in x of degree n.

2. Prove that the system of exponential functions in equation (4) of this section is orthogonal on the interval $(-\pi, \pi)$ in the Hermitian sense.

3. Prove that the system $\Big\{\exp\Big(\frac{}{b - a}$ $0, \pm 1, \pm 2,$ $),$ where $\exp(u)$ denotes e^u, is orthogonal in the Hermitian sense on the interval (a, b).

24. Orthogonal Functions Generated by Differential Equations. In solving the problem of displacements in a stretched string in Sec. 13, we used particular solutions of the partial differential equation of motion which vanished when $x = 0$ and $x = L$. In order that $y = X(x)T(t)$ be such a solution, it was found that the function $X(x)$ must satisfy the conditions

$$X''(x) + \lambda X(x) = 0, \tag{1}$$

$$X(0) = 0, \qquad X(L) = 0, \tag{2}$$

for some constant value of λ, denoted there by $-\gamma$.

Equations (1) and (2) form a homogeneous boundary value problem in ordinary differential equations containing λ as a parameter. Since the solution of equation (1) that vanishes when $x = 0$ is $X = C \sin \sqrt{\lambda} x$, the problem has solutions not identically zero only if λ satisfies the equation

$$\sin \sqrt{\lambda} L = 0. \tag{3}$$

Therefore $\lambda = n^2\pi^2/L^2$ $(n = 1, 2, \cdots)$, and the corresponding solutions of the problem (1)-(2) for these values of λ are,

except for a constant factor, sin $(n\pi x/L)$. These functions were shown to form an orthogonal set on the interval $(0, L)$.

Corresponding results can be found in much more general cases. When applied to a more general partial differential equation, separation of variables will yield an equation in $X(x)$ of the type

$$X'' + f_1(x)X' + [f_2(x) + \lambda f_3(x)]X = 0.$$

Here f_1, f_2, and f_3 are known functions involved in the coefficients of the partial differential equation, and λ is the constant which arises upon separation of variables.

When the last equation is multiplied through by the factor $r(x)$, where

$$r(x) = e^{\int f_1(x)\,dx},$$

it takes the form

(4)

known as the *Sturm-Liouville equation.*

The boundary conditions on $X(x)$ may have the form

(5) $$(a) + a_2X'(a) = 0, \qquad b_1X(b) + b_2X'(b) = 0,$$

where a_1, a_2, b_1, and b_2 are constants.

The problem composed of the differential equation (4) and the boundary conditions (5) is called a *Sturm-Liouville problem* or *system,* in honor of the two mathematicians who made the first extensive study of that problem.*

Under rather general conditions on the functions p, q, and r, it can be shown that there is a discrete set of values $\lambda_1, \lambda_2, \cdots$ of the parameter λ for which the system (4)-(5) has solutions not identically zero. These numbers λ_n are called the *characteristic numbers* of the system. In the above special case—equations (1) and (2)—they are the numbers $n^2\pi^2/L^2$, the roots of the characteristic equation (3).

The solutions $X_n(x)$ $(n = 1, 2, \cdots)$, obtained when $\lambda = \lambda_n$ in equation (4), are the *characteristic functions* of the Sturm-Liouville problem. These are the functions sin $(n\pi x/L)$ in the special case.

* Papers by Liouville and Sturm on this problem will be found in the first three volumes of *Journal de mathématique,* 1836–1838.

It will be shown in the following section that the set of fu tions $\{X_n(x)\}$ $(n = 1, 2, \cdots)$ is orthogonal on the inte (a, b) with respect to the weight function $p(x)$.

Moreover, it can be shown that any function $f(x)$, definec the interval (a, b) and satisfying certain restrictions as to continuity and differentiability, is represented by its generali Fourier series corresponding to that orthogonal set of functio That is, if $\varphi_n(x)$ is the function obtained by normalizing $X_n($ the series

(6)

where

converges to the function $f(x)$ in the interval (a, b). It shoul be noted that the normalizing factor for X_n here is the numbe

When $r(b) = r(a)$, the statements made above are also tru when the boundary conditions (5) are replaced by the condition

(7) $X(a)$ $X'(a) =$

called the *periodic boundary conditions.* Conditions of this sort frequently arise when x represents a coordinate such as the angle θ in polar coordinates, or $\cos\theta$.

The proof that series (6) converges to the function $f(x)$ is quite long and involved, as we may well expect in view of the fact that the coefficients in differential equation (4) are arbitrary functions of x. The proofs generally make use of the theory of functions of complex variables, or the comparison of the expansion with a Fourier series, or both. The development of a general expansion theorem, along with other interesting and useful results in the general theory of Sturm-Liouville systems, is beyond the scope of the present volume.*

The expansions considered in the chapters to follow are all special cases of the general theory. But in two of the important cases, those of Bessel and Legendre functions, the equations are

* A treatment of these topics is included in a companion volume, now being prepared by the author, on further methods of solving partial differential equations. Also see Ince, "Ordinary Differential Equations," pp. 235 *ff.*, 1927; and the references listed at the end of the present chapter.

singular cases of equation (4) which must be treated separately even in the general theory. Hence the plan of presentation followed here is not especially inefficient.

Many of the important sets of orthogonal functions are generated in the above manner, as solutions of a homogeneous differential system involving a parameter. The expansion theorem shows that these sets are closed with respect to ordinary convergence, rather than convergence in the mean, so that we have an important advantage over the general theory discussed in the preceding sections.

25. Orthogonality of the Characteristic Functions. Two theorems from the general theory of Sturm-Liouville systems can easily be established here. They will be useful in the following chapters. The first shows the orthogonality of the characteristic functions, and the second shows that the characteristic numbers are real. The existence of such functions and numbers will be established in each case treated later on, of course, by actually finding them.

Theorem 3. *Let the coefficients p, q, and r in the Sturm-Liouville problem be continuous in the interval $a \leqq x \leqq b$, and let λ_m, λ_n be any two distinct characteristic numbers, and $X_m(x)$, $X_n(x)$ the corresponding characteristic functions, whose derivatives $X_m'(x)$, $X_n'(x)$ are continuous. Then $X_m(x)$ and $X_n(x)$ are orthogonal on the interval (a, b), with respect to the weight function $p(x)$.*

Furthermore, in case $r(a) = 0$, the first of the conditions (5), *Sec.* 24, *can be dropped from the problem, and if $r(b) = 0$ the second of those conditions can be dropped. If $r(b) = r(a)$, those conditions can be replaced by the periodic conditions* (7), *Sec.* 24.

Since X_m and X_n are solutions of equation (4), Sec. 24, when $\lambda = \lambda_m$ and $\lambda = \lambda_n$, respectively,

$$d \qquad \lambda_m p)X_m = 0,$$

$$\frac{d}{dx}(rX_n') + (q + \lambda_n p)X_n = 0.$$

Multiplying the first equation by X_n and the second by X_m, and subtracting, gives

$$(\lambda_m - \qquad = X_m \frac{d}{\quad} \qquad) - X \quad d$$

$$= \frac{d}{dx}[(rX_n')X_m -$$

Integrating both members over the interval (a, b),

$$(1) \qquad (\lambda_m - \lambda_n) \int_a^b pX_mX_n\,dx = \Big[r(X_mX_n' - X_nX_m')\Big]_a^b.$$

In the special case when $a_2 = b_2 = 0$, the boundary conditions (5), Sec. 24, become

$$(2) \qquad X(a) = 0, \qquad X(b) = 0.$$

Since both $X_m(x)$ and $X_n(x)$ then satisfy these conditions, it is evident that the right-hand member of equation (1) vanishes. But $\lambda_m - \lambda_n \neq 0$, so that

(3)

which is the statement of orthogonality between X_m and X_n. In case $r(a) = 0$, it is clear that (3) follows from (1) without the use of the first of the conditions (2). Similarly, if $r(b) = 0$, the second condition is not needed.

The proof that equation (3) follows when the general boundary conditions (5), Sec. 24, or the periodic boundary conditions, are substituted for (2), will be left for the problems.

Theorem 4. *If in addition to the conditions stated in Theorem 3 the coefficient $p(x)$ does not change sign in the interval (a, b), then every characteristic number of the Sturm-Liouville problem is real.**

Suppose there is a complex characteristic number λ, where

$$\lambda = \alpha + i\beta.$$

Let

$$X(x) = u(x) + iv(x)$$

be the corresponding characteristic function. Substituting this in the Sturm-Liouville equation, we have

$$\frac{d}{dx}(ru' + irv') + (q + \alpha p + i\beta p)(u + iv) = 0.$$

Equating the real and imaginary parts to zero, separately,

$$\frac{d}{dx}(ru') + (q + \alpha p)u - \beta pv = 0,$$

$$\frac{d}{dx}(rv') + (q + \alpha p)v + \beta pu = 0,$$

* The functions p, q, and r are assumed to be real.

and, upon multiplying the first of these equations by v and the second by u, and subtracting, it follows that

$$-\beta(u^2 + v^2)p = u\frac{d}{dx}(rv') - v\frac{d}{dx}(ru')$$
$$= \frac{d}{dx}[(rv')u - (ru')v].$$

Consequently, an integration gives the relation

$$(4) \qquad -\beta\int_a^b (u^2 + v^2)p\,dx = (uv' - u'v)r\Big]_a^b.$$

Again let us complete the proof here when the boundary conditions of the Sturm-Liouville problem have the special form (2). Since our characteristic function $u + iv$ satisfies (2), its real and imaginary parts must each vanish when $x = a$ and $x = b$. The right-hand member of (4) therefore vanishes. But if the function $p(x)$ in the integral does not change sign in the interval $(a; b)$, the integrand itself cannot change sign, and so the integral cannot vanish. It follows that $\beta = 0$, and therefore the characteristic number λ is real.

As before, if $r(a) = 0$, the first of conditions (2) is not needed, and if $r(b) = 0$, the second can be dropped.

The argument is not essentially different when the more general boundary conditions (5), Sec. 24, or the periodic conditions are used. This matter is left for the problems.

PROBLEMS

1. Complete the proof of Theorem 3 when the boundary conditions are (a) the conditions (5), Sec. 24; (b) the periodic conditions (7), Sec. 24, assuming that $r(b) = r(a)$.

2. Complete the proof of Theorem 4 when the boundary conditions are (a) the conditions (5), Sec. 24; (b) the periodic conditions (7), Sec. 24, assuming that $r(b) = r(a)$.

Find the characteristic functions of each of the following special cases of the Sturm-Liouville problem. Also note the interval and weight function in the orthogonality relation ensured by Theorem 3, and find the normalizing factors.

3. $X'' + \lambda X = 0$; $X'(0) = 0$, $X'(L) = 0$.

Ans. $X_n = \cos(n\pi x/L)$ $(n = 0, 1, 2, \cdots)$.

4. $X'' + \lambda X = 0$; $X(0) = 0$, $X'(L) = 0$.

Ans. $\varphi_n = \sqrt{2/L}\sin[(2n - 1)\pi x/(2L)]$ $(n = 1, 2, \cdots)$.

5. $X'' + \lambda X = 0$; $X(\pi) = X(-\pi)$, $X'(\pi) = X'(-\pi)$.

Ans. $\{1, \cos x, \cos 2x, \cdots, \sin x, \sin 2x, \cdots\}$.

6. $X'' + \lambda X = 0$; $X(0) = 0$, $X'(1) + hX(1) = 0$, where h is a constant. Show in this case that $X_n = \sin \alpha_n x$, where α_n represents the positive roots of the equation $\tan \alpha = -\alpha/h$, an equation whose roots can be approximated graphically. Also show that X_n is normalized by multiplying it by $\sqrt{2h/(h + \cos^2 \alpha_n)}$.

7. $(d/dx)(x^3X') + \lambda xX = 0$; $X(1) = 0$, $X(e) = 0$. Note that the equation here reduces to one of the Cauchy type after the indicated differentiation is carried out.

Ans. $\varphi_n = (\sqrt{2}/x) \sin (n\pi \log x)$ $(n$ 1, 2.

REFERENCES

1. Courant, R., and D. Hilbert: "Methoden der mathematischen Physik," Vol. 1, 1931.
2. Mises, R. v.: "Die Differential- und Integralgleichungen der Mechanik und Physik" (Riemann-Weber), Vol. 1, 1925.

CHAPTER IV

FOURIER SERIES

26. Definition. The trigonometric series

$$(1)\quad \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots$$

is a *Fourier series* provided its coefficients are given by the formulas

$$(2)\quad \begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad && (n = 0, 1, 2, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad && (n = 1, 2, \end{aligned}$$

where $f(x)$ is some function defined in the interval $(-\pi, \pi)$. In particular, series (1) with the coefficients (2) is called the *Fourier series corresponding to* $f(x)$ *in the interval* $(-\pi, \pi)$, written

$$(3)\qquad + \qquad \cos nx + b_n \sin \qquad (-\pi < x < \pi).$$

Formulas (2) for the coefficients are special cases of those for the generalized Fourier series in the chapter preceding. The functions 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$ constitute an orthogonal (but not normalized) set in $(-\pi, \pi)$. This was noted in Prob. 5, Sec. 25; but we can easily show it here independently. For if $m, n = 0, 1, 2, \cdots$, then

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0,$$
$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0, \qquad \text{if } m \neq n,$$

and whether m and n are distinct or not,

$$\int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0.$$

When $m = n$ the first two integrals become

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi \qquad \text{if } n \neq 0,$$
$$= 2\pi \qquad \text{if } n = 0;$$
$$\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi.$$

Considering (3) as an equality and multiplying first by $\cos nx$ and integrating therefore gives formally

$$\cos nx\, dx = \pi a_n \quad (n = 0, 1, 2, \cdot \quad \cdot).$$

Similarly, multiplying by $\sin nx$ and integrating gives

$$\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \pi b_n \quad (n = 1, 2, \cdot$$

These are formulas (2) for the Fourier coefficients.

Again, the corresponding orthonormal set of functions is

$$1 \quad \frac{\cos x}{\pi}, \frac{\sin x}{\sqrt{\pi}}, \cos 2x \quad \sin 2x$$

and the Fourier constants c_n of $f(x)$ corresponding to these functions are the integrals of the products, or the inner products, of these functions by $f(x)$. So the Fourier series, with respect to this set, corresponding to $f(x)$, is

$$f(x')\, dx'$$

$$\cos nx$$

$$\int_{-\pi}^{\pi} \quad x') \frac{\sin nx'}{\sqrt{\pi}} dx' \frac{\sin nx}{\sqrt{\pi}}\Bigg],$$

where x' is used for the variable of integration. This can be written

$$(4) \quad f(x) \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x') \quad + \frac{1}{\pi} \quad \Bigg[\cos nx \int_{-\pi}^{\pi} f(x') \cos nx'\, dx'$$
$$+ \sin nx \int_{-\pi}^{\pi} f(x') \sin nx'\, dx'\Bigg];$$

ich is the same as series (3) with the coefficients (2).

Formula (4) can be written in the more compact form

(5)

$$\frac{1}{\pi}\sum_{1}^{\infty}\int_{-\pi}^{\pi} \qquad \cos$$

Note that the constant term is the mean value of $f(x)$ over the interval $(-\pi, \pi)$.

27. Periodicity of the Function. Example. Every term in the above series is periodic with the period 2π. Consequently, if the series converges to $f(x)$ in the interval $(-\pi, \pi)$, it must converge to a periodic function with period 2π for all values of x. Thus it would represent $f(x)$ for every finite value of x, provided

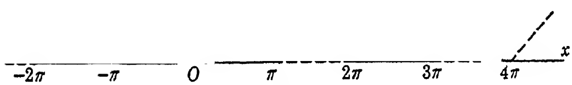

FIG. 6.

the definition of $f(x)$ is extended to include all values of x by the periodicity relation

$$f(x + 2\pi) = f(x).$$

Thus *the Fourier series may conceivably serve either of two purposes:* (*a*) to represent a function defined in the interval $(-\pi, \pi)$, for values of x in that interval, or (*b*) to represent a periodic function, with period 2π, for all values of x. It clearly cannot represent a function for all values of x if that function is not periodic.

The particular interval $(-\pi, \pi)$ was introduced only as a matter of convenience. We shall soon see that it is easy to change to any other finite interval.

It is not necessary that $f(x)$ be described by a single analytic expression, or that it be continuous, in order to determine the coefficients in its Fourier series. Of course the mere fact that the series can be written does not ensure its convergence or, if convergent, that its sum will be $f(x)$. Conditions for this are to be established in Sec. 33.

Example. Write the Fourier series corresponding to the function $f(x)$ defined in the interval $-\pi < x < \pi$ as follows:

$$
\begin{aligned}
f(x) &= 0 \qquad \text{when } -\pi < x \leqq 0,\\
&= x \qquad \text{when } 0 \leqq x < \pi.
\end{aligned}
$$

The graph of this function is indicated by the heavy lines in Fig. 6. The Fourier coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0 + \int_0^{\pi} x\,dx\right) = \frac{\pi}{\cancel{2}},\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx\\
&= \frac{1}{\pi n^2}\Big[\cos nx + nx\sin nx\Big]_0^{\pi} = \frac{1}{\pi n^2}(\cos n\pi - 1),\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx\\
&= \frac{1}{\pi n^2}\Big[\sin nx - nx\cos nx\Big]_0^{\pi} = -\frac{\cos n\pi}{\;}
\end{aligned}
$$

The series is therefore

(1)

$$
\begin{aligned}
&\frac{\pi}{4} + \left(\sin x - \frac{2}{\pi}\cos x\right) - \frac{1}{2}\sin 2x\\
&\quad + \left(\frac{1}{3}\sin 3x - \frac{2}{9\pi}\cos 3x\right) - \frac{1}{4}\sin 4x +
\end{aligned}
$$

If this converges to $f(x)$ when $-\pi < x < \pi$, it also converges for all other values of x to the periodic function represented by the dotted lines in the figure. Note that this periodic function is discontinuous at $x = \pm\pi, \pm 3\pi, \cdots$; the value represented by the series at such points will be found later.

As an indication of the convergence of the series (1) to $f(x)$ it is instructive to sum a few terms of the series by composition of ordinates. It will be found, for instance, that the graph of the curve

$$
y = \frac{\pi}{4} + \sin x - \frac{2}{\pi}\cos x - \frac{1}{2}\sin 2x
$$

is a wavy approximation to the curve shown in the figure. The addition of more terms from the series generally improves the approximation.

PROBLEMS

Write the Fourier series corresponding to each of the following functions defined for $-\pi < x < \pi$. In a few of the problems, sum a few terms of the series graphically; also show graphically the periodic function which is represented by the series provided the series converges to the given function.

1. $f(x) = x$ when $-\pi < x < \pi$. (Also note the sum of the series when $x = \pm\pi$.)
Ans. $2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin nx.$

2. $f(x) = e^x$ when $-\pi < x < \pi$.
Ans. $\frac{2 \sinh \pi}{\pi}\left[\frac{1}{2} + \sum_1^\infty \frac{(-1)^n}{1+n^2}(\cos nx - n \sin nx)\right].$

3. $f(x$ 1 when $-\pi < x < 0$; $f(x) = 2$ when $0 < x < \pi$.

sin nx.

$f(x) = 0$ when $-\pi < x < 0$; $f(x) = \sin x$ when $0 < x < \pi$.
Ans. $\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum^\infty$ cos $2nx$

28. Fourier Sine Series. Cosine Series. When $f(-x) = -f(x)$, $f(x)$ is called an *odd* function; its graph is symmetric with respect to the origin, and its integral from $-\pi$ to π is zero. When $f(-x) = f(x)$, the function is *even;* its graph is symmetric to the axis of ordinates, and

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_0^\pi f(x)\,dx.$$

As examples, the functions x, x^3, and $x^2 \sin kx$ are odd, while 1, x^2, $\cos kx$, and $x \sin kx$ are even.

Although most functions are neither even nor odd, every function can be written as the sum of an even and an odd one by means of the identity

$$= \tfrac{1}{2}[f(x) \quad (-x)] + \quad - f(-x)].$$

When $f(x)$ is an odd function defined in $(-\pi, \pi)$, formulas (2), Sec. 26, for its Fourier coefficients become

$$a_n = 0 \qquad (n = 0, 1, 2, \cdots),$$
$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx \qquad (n = 1, 2, \cdots).$$

Hence its Fourier series reduces to

$$(2) \qquad f(x) \sim \frac{2}{\pi} \sum_1^\infty \sin nx \int_0^\pi f(x') \sin nx' \, dx'.$$

The series in (2) is known as the *Fourier sine series*. It can clearly be written when $f(x)$ is any function defined in the interval $(0, \pi)$, provided the integrals representing its coefficients exist. Furthermore, when $f(x)$ is defined only in the interval $(0, \pi)$, an odd function exists in $(-\pi, \pi)$ which is identical with $f(x)$ in $(0, \pi)$. If that odd function is represented by its Fourier series, so is $f(x)$ in $(0, \pi)$. Thus the question of convergence of the sine series to $f(x)$ in $(0, \pi)$ depends directly upon the conditions of convergence of the series in the last section.

Similarly, when $f(x)$ is an even function defined in the interval $(-\pi, \pi)$, the coefficients in its Fourier series are

$$= \frac{2}{\pi} \int_0^\pi J \qquad \cos nx \, dx \qquad (n = 0, 1, 2, \cdots),$$
$$(n = 1, 2, \cdots);$$

and the series becomes the *Fourier cosine series*

$$(3) \quad f(x) \sim \frac{1}{\pi} \int_0^\pi f(x') \, dx' + \frac{2}{\pi} \sum_1^\infty \cos nx \int_0^\pi f(x') \cos nx' \, dx'.$$

When $f(x)$ is defined only in $(0, \pi)$, this series can be written, in general, and again the conditions under which it converges to $f(x)$ will be known when the conditions are found for the more general series in the last section.

For functions defined in the interval $(0, \pi)$, then, both the sine series and the cosine series representations can be considered. As indicated earlier, these are the series corresponding to $f(x)$ with respect to two different sets of functions, $\{\sqrt{2/\pi} \sin nx\}$, and $\{1/\sqrt{\pi}, \sqrt{2/\pi} \cos nx\}$ $(n = 1, 2, \cdots)$, each of which is

orthonormal in $(0, \pi)$. The series (2) and (3) can be written more easily from this viewpoint; but in the theory of these series it is important to consider them as special cases of the series in Sec. 26.

Every term of the sine series is an odd function. So if the series converges when $0 < x < \pi$, it must converge to an odd function with the period 2π for all values of x.

Similarly, if the cosine series converges, it must represent an even periodic function with period 2π.

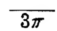

FIG. 7.

29. Illustration. Let us write (*a*) the Fourier sine series, and (*b*) the Fourier cosine series, corresponding to the function $f(x)$, defined in the interval $0 < x < \pi$ as follows:

$$f(x) = x \qquad \text{when } 0 \leqq x < \frac{\pi}{2},$$

$$= 0 \qquad \text{when } \frac{\pi}{2} < x \leqq \pi.$$

a. The coefficients in the sine series are

$$b_n = \frac{2}{\pi}\int_0 f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x \sin nx \, dx$$

so the series is

$$1 \sum^{\infty} / 2 \qquad \frac{n\pi}{2} - \frac{\pi}{n}\cos\frac{n\pi}{2} \qquad nx$$

$$= \frac{1}{\pi}\left(2 \sin x + \frac{\pi}{2}\sin 2x - \frac{2}{9}\sin 3x - \frac{\pi}{4}\sin 4x + \qquad\right).$$

If this sine series converges to our function $f(x)$, it must also represent the odd periodic extension of $f(x)$ shown in Fig. 7.

b. The coefficients in the cosine series are

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0 \quad x\,dx = \frac{\pi}{4},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x \cos$$

$$= \frac{1}{\pi n^2}\left(2\cos\frac{n\pi}{2} + n\pi \sin\frac{n\pi}{2} - 2\right)$$

Therefore

$$= \frac{\pi}{8} + \frac{1}{\pi}\Big[\Big(\qquad \cos x - \cos \qquad - \left(\frac{\pi}{3} + \frac{2}{9}\right)\cos \qquad \Big]$$

Assuming its convergence to $f(x)$, this cosine series would converge for all x to the even periodic function shown in Fig. 8.

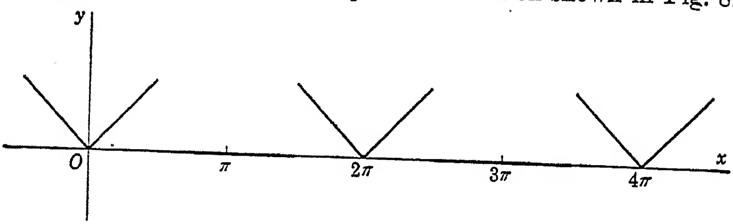

FIG. 8.

PROBLEMS

Find (*a*) the Fourier sine series, and (*b*) the Fourier cosine series, corresponding to each of the following functions defined in the interval $(0, \pi)$. Assuming that each series represents its function within that interval, show what function it represents outside the interval.

1. $f(x) = x$ when $0 < x < \pi$. (Compare Prob. 1, Sec. 27.)

∞

s. (*a*) $\qquad \cos\,(2n-1)x$ / n

2. $f(x) = \sin x$ when $0 < x < \pi$.

Ans. (*a*) sin

3. $f(x) = \cos x$ when $0 < x < \pi$.

Ans. (*a*) $\frac{8}{\pi}$ $\qquad \frac{\sin 2nx}{\quad - 1}$; (*b*) $\cos x$.

) $= \pi - x$ when $0 < x < \pi$.

Ans. (a) $2 \sum_1^\infty \frac{\sin nx}{n}$; (b) $\frac{\pi}{2} + \frac{\pm}{\pi} \sum_1 \frac{\cos (2n-1)}{(2n-1)^2}$

5. $f(x) = 1$ when $0 < x < \pi/2$, $f(x) = 0$ when $\pi/2 < x < \pi$.

$$\frac{2}{\pi} \sum_1 \left(1 - \cos \frac{n\pi}{2}\right) \sin$$

$$\cos \qquad \frac{}{2n-1}$$

1

6. $f(x) = x$ when $0 < x < \pi/2$, $f(x) = \pi - x$ when $\pi/2 < x < \pi$.

Ans. (a) (Compare Sec. 14); $\pi \quad 8 \cdot \sum_1 \frac{\cos (4n-2)x}{(4n-2)^2}.$

7. $f(x) = e^x$ when $0 < x < \pi$.

Ans. 2 $\sin nx$

(b) $\cos$

8. Obtain series (4), Sec. 26, for any function in $(-\pi, \pi)$ from series (2) and (3), Sec. 28, for odd and even functions, respectively, using identity (1), Sec. 28.

30. Other Forms of Fourier Series. The Fourier series corresponding to any function $F(z)$, defined in the interval

$$-\pi < z < \pi,$$

is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(z')\, dz' + \frac{1}{\pi} \sum_1^\infty \left[\cos nz \qquad \cos nz' \right.$$

$$\left. + \sin nz \int_{-\pi}^{\pi} F(z') \sin nz'\, dz' \right].$$

Substituting the new variable x and the new variable of integration x' throughout, where

$$x = \frac{Lz}{\pi}, \qquad \frac{Lz'}{\pi},$$

and writing $f(x)$ for $F(\pi x/L)$, the above correspondence becomes

$$\sin\frac{n\pi x}{L}\int_{-L}^{L} f(x')\sin\frac{n\pi x'}{L}$$

The series in (1) is the *Fourier series on the interval* $(-L, L)$ corresponding to any function $f(x)$ defined in that interval.

The same substitution changes the sine series to one corresponding to a function $f(x)$ defined in the interval $(0, L)$, or an odd function in $(-L, L)$:

$$(2) \qquad f(x) \sim \frac{2}{L}\sum_{1} \sin \qquad \int_0^L f(x')\sin\frac{n\pi x'}{L}\,dx'.$$

It also changes the cosine series to the form

$$(3) \quad f(x) \quad \frac{1}{L}\int_0^L f(x')\,dx \qquad \cos \qquad \cos\frac{n\pi x'}{L}\,dx',$$

corresponding to a function $f(x)$ defined in the interval $(0, L)$, or an even function in $(-L, L)$. The substitution simply changes the unit of length on the x-axis.

Of course the forms (1) to (3) can also be written by noting the orthogonality of the sine and cosine functions involved there in the interval $(-L, L)$ or $(0, L)$. Let us obtain the series for any interval (a, b) in this manner.

Upon integrating, it will be found that

$$\int_a^b \exp \qquad \left(\frac{2n\pi i x}{a}\right) dx = 0 \qquad \text{if } m \neq -n,$$
$$= b - a \quad \text{if } m = -n.$$

That is, the set of complex functions

is orthogonal on the interval (a, b), in the Hermitian sense.

Assuming a series representation of $f(x)$ in terms of those functions,

$$f(x) = \quad C_n \exp \qquad (a < x < b),$$

the coefficients C_n can be found formally by multiplying through by $\exp[-2m\pi i x/(b - a)]$ and integrating. In view of the above orthogonality property, this gives

$$\exp \quad -\frac{2m\pi i x}{b - a}\Big) dx = (b - a)$$

Thus, the *exponential form* of the Fourier series corresponding to a function $f(x)$ defined in the interval (a, b) is

$$\frac{1}{-a} \sum^{\infty}$$

Grouping the terms for which the indices n differ only in sign, (4) takes the *trigonometric form*,

$$(5) \quad f(x) \qquad b -$$

$$\frac{2n\pi(x' - x)}{b - a} dx',$$

of the Fourier series *corresponding to* $f(x)$ *in* (a, b). This can be obtained as well from the earlier form (5), Sec. 26, for the interval $(-\pi, \pi)$ by making a linear substitution in the variables x and x'.

These additional forms of the series, therefore, arise from the original form for the interval $(-\pi, \pi)$ by changing the origin and the unit of length on the x-axis. So it is only necessary to develop the theory of convergence of the series for the interval $(-\pi, \pi)$; the results will then be evident for the other forms.

Form (5) contains the earlier forms as special cases. The series represents a periodic function with period $(b - a)$, if it converges. Therefore it can be considered as a possible expansion of either a function which is periodic with period $(b - a)$, or a function which is defined only in the interval (a, b). Both types of applications are important. In the second case,

however, there may be many Fourier series representations of the function; for the function can be defined at pleasure in any extension of the interval, and the new series in the extended interval may still represent its function. It would then represent the given function in (a, b).

PROBLEMS

1. Write formula (1) in the form corresponding to (3)-(2), Sec. 26; also in the form corresponding to (5), Sec. 26.

2. Write formula (5) when $a = 0$, $b = 2L$, and compare it with formula (1).

Write the Fourier series corresponding to each of the following functions.

3. $f(x) = -1$ when $-L < x < 0$, $f(x) = 1$ when $0 < x < L$.

$$Ans.\ \frac{4}{\pi} \qquad \frac{\quad}{2n-1} \sin \frac{(2n -}{}$$

4. $f(x) = |x|$ when $-L < x < L$; that is, $f(x) = -x$ in $(-L, 0)$ and $f(x) = x$ in $(0, L)$. $\quad Ans.\ \frac{L}{2} - \frac{4L}{\pi^2} \qquad \frac{\quad}{(2n-1)^2} \cos \frac{(2n -}{}$

5. $\quad x^2$ when $-L < x < L$.

$$L^2$$

6. $f(x) = x + x^2$ when $-1 < x < 1$.

$$\frac{1}{3} + \frac{2}{\pi} \sum^{\infty} \qquad \cos \qquad - \frac{1}{n} \sin n\pi x\Big).$$

7. $f(x) = 0$ when $-2 < x < 1$, $f(x) = 1$ when $1 < x < 2$.

$$\frac{1}{4} \quad 1 \sum^{\infty} 1 \qquad \frac{n\pi}{2} \cos \frac{n\pi x}{2} + \qquad - \cos \frac{n\pi}{2}\Big) \sin \frac{n\pi x}{2}$$

8. $f(x) = 1$ when $0 < x < 1$, $f(x) = 2$ when $1 < x < 3$, and $f(x + 3) = f(x)$ for all x.

$$\frac{2\pi n}{3}\Big) \sin \frac{\quad}{3}\Big]$$

9. $f(x) = e^x$ when $0 < x < 1$, using the exponential form of the Fourier series.

31. Sectionally Continuous Functions. At this point let us introduce some special classes of functions, the use of which will

keep the theory which is to follow on a fairly elementary level. These classes will include most of the functions which arise in the applications; but they are rather old-fashioned classes. As we shall point out from time to time, our principal results can be obtained for a considerably broader class of functions by using somewhat more advanced methods of modern analysis.

A function is *sectionally continuous*, or *piecewise continuous*, in a finite interval if that interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite limits as the variable approaches either end point from the interior. Any discontinuities of such a function are of the type known as *ordinary* points of discontinuity. Every such function is bounded and integrable over the interval, its integral being the sum of a finite number of integrals of continuous functions.

The symbol $f(x_0 + 0)$ denotes the limit of $f(x)$ as x approaches x_0 from the right. For $f(x_0 - 0)$ the approach is from the left. That is, if λ is positive,

$$f(x_0 + 0) = \lim_{\lambda \to 0} f(x_0 + \lambda),$$
$$f(x_0 - 0) = \lim_{\lambda \to 0} f(x_0 - \lambda).$$

We define the *right-hand derivative*, or *derivative from the right*, of $f(x)$ at x_0 as the following limit:

$$\lambda \to 0 \qquad\qquad \lambda$$

where λ is positive, provided of course that this limit exists. Similarly, the *left-hand derivative* is

where λ is again a positive variable.

It follows at once that if $f(x)$ has an ordinary derivative $f'(x)$ at x_0, then its derivatives from the right and left both exist there and have the common value $f'(x_0)$. But a function may have one-sided derivatives without having an ordinary derivative. For example, if

$$\begin{aligned} f(x) &= x^2 && \text{when } x \leqq 0, \\ &= \sin x && \text{when } x \geqq 0, \end{aligned}$$

then $f'(0)$ does not exist, but at the point $x = 0$ the derivatives from the right and left have the values 1 and 0, respectively. Again, for the step function

$$
\begin{aligned}
f(x) &= 0 \qquad \text{when } x < 0, \\
&= 1 \qquad \text{when } x > 0,
\end{aligned}
$$

$f'(0)$ does not exist, but its one-sided derivatives have the common value zero.

All the functions described in the problems and examples in this book are sectionally continuous and have one-sided derivatives.

If two functions $f_1(x)$ and $f_2(x)$ have derivatives from the right at a point $x = x_0$, so does their product. For the right-hand derivative of their product is the limit, as λ approaches zero through positive values, of the ratio

$$\lambda)f_2(x_0 + \lambda) - \qquad 0)$$

This can be written

The limit of $f_1(x_0 + \lambda)$ exists, and the limits of the two fractions exist, since they represent the right-hand derivatives of $f_2(x)$ and $f_1(x)$ at the point x_0. Hence the limit of the ratio representing the right-hand derivative of the product f_1f_2 exists.

In the same manner it can be seen that the left-hand derivative of the product exists at each point where the two factors have left-hand derivatives.

One further property will be useful, in connection with our theorem on the differentiation of Fourier series (Chap. V).

Let $f(x)$ be a function which is continuous in an interval $a \leqq x \leqq b$, and whose derivative $f'(x)$ exists and is continuous at all interior points of that interval. Also let the limits $f'(a + 0)$ and $f'(b - 0)$ exist. Then the right-hand derivative of $f(x)$ exists at $x = a$, and the left-hand derivative exists at $x = b$, and these have the values $f'(a + 0)$ and $f'(b - 0)$, respectively.

Since $f(x)$ is continuous, and differentiable when $a < x < b$, the law of the mean applies. So, for every λ $(0 < \lambda < b - a)$, a number θ $(0 < \theta < 1)$ exists such that

Since $f'(a + 0)$ exists, the limit, as λ approaches zero, of the function on the right exists and has that same value. The function on the left must have the same limit; that is, the derivative from the right at $x = a$ has the value $f'(a + 0)$.

Similarly for the derivative from the left at $x = b$.

It follows at once that *if $f(x)$ and $f'(x)$ are sectionally continuous, the one-sided derivatives of $f(x)$ exist at every point.*

32. Preliminary Theory. In order to establish conditions under which a Fourier series converges to its function, a few preliminary theorems, or lemmas, on limits of trigonometric integrals are useful. The integrals involved in these lemmas are known as Dirichlet's integrals.

The lemmas here will be so formulated that they can also be used in the theory of the Fourier integral (Chap. V). There it is essential that the parameter k used in the first lemma be permitted to vary continuously rather than just through the positive integers. In the latter case $(k = n)$ the limit in Lemma 1 would follow quite easily from equation (6), Sec. 21.

Lemma 1. *If $F(x)$ is sectionally continuous in the interval*

$$\lim_{k\to\infty} \int_a^b F(x) \sin kx \, dx = 0. \tag{1}$$

Lemma)

Let the interval (a, b) be divided into a finite number of parts in each of which $F(x)$ is continuous, and let (g, h) represent any one of those parts. Then, if it is shown that

$$\lim_{k\to\infty} \int_g^h F(x) \sin kx \, dx = 0, \tag{2}$$

the lemma will be proved.

Divide the interval (g, h) into r equal parts by the points $x_0 = g, x_1, x_2, \cdots, x_r = h$. Then the integral in equation (2) can be written

$$\quad^{r-1} \qquad ^{x_{i+1}} F(x) \sin kx \, dx,$$

or

$$r-1 \qquad (x_i) \sin kx\, dx \qquad \sin$$

Carrying out the first integration and using the fact that the absolute value of an integral is not greater than the integral of the absolute value of the integrand, we find that

$$(3) \quad \left| \int_g^h F(x) \sin kx\, dx \right. \qquad r-1 \qquad F(x_i)\, \cos kx_i - \cos$$

$$\left. \int_{x_i}^{x_{i+1}} |[F(x) - F(x_i)] \sin kx|\, dx \right\}.$$

The oscillation of $F(x)$ in the interval (x_i, x_{i+1}) is the difference between the greatest and least values of the function in that interval. Let η_r be the greatest oscillation of $F(x)$ in any of the r intervals (x_i, x_{i+1}), so that $|F(x) - F(x_i)| \leqq \eta_r$ in each interval. Also let M be the greatest value of $|F(x)|$ in the interval (g, h). Then according to (3),

$$F(x) \sin kx\, dx \qquad \left[\frac{2M}{k} \right.$$

Now let r be selected as the largest integer which does not exceed $\sqrt{k}$. Then

and this approaches zero as k tends to infinity. But r tends to infinity with k, and so η_r approaches zero too, because the oscillation of a continuous function approaches zero uniformly in all intervals of length $(h - g)/r$ as r becomes infinite. Hence

$$\lim_{k \to \infty} \left| \int_g^h F(x) \sin kx\, dx \right. = 0;$$

so relation (2) is true, and the lemma is established.

Lemma 2. *If $F(x)$ is sectionally continuous in the interval $0 \leqq x \leqq b$ and has a right-hand derivative at $x = 0$, then*

(4) lim

The integral in (4) can be written as the sum

$$\int_0^b \qquad \frac{\cdots}{x}\, dx + \int_0^b \frac{\qquad}{x} \qquad \sin kx\, dx.$$

Consider the first of these integrals. We can write

$$\lim_{k \to \infty} \int_0^b \qquad \frac{\sin kx}{\cdot}\, dx = \qquad \lim_{k \to \infty} \int_0^{kb} \frac{\sin u}{u}\, du = \frac{\pi}{2}$$

since

sin

The function $[F(x) - F(+0)]/x$ in the second integral in (5) is sectionally continuous in the interval $(0, b)$ since $F(x)$ itself is, and since

exists because $F(x)$ has a right-hand derivative at $x = 0$. Lemma 1 therefore applies to the second integral in (5), giving

$$\lim_{k \to \infty} \int_0^b \qquad - F(+0) \quad \text{sin}$$

The limit of expression (5) is therefore $F(+0)\pi/2$; hence (4) is true and the lemma is proved.

Lemma 3. *If $F(x)$ is sectionally continuous in the interval (a, b) and has derivatives from the right and left at a point $x = x_0$, where $a < x_0 < b$, then*

$$(6) \qquad \lim_{k \to \infty} \int_a^b F(x) \frac{\sin k(x - x_0)}{x - x_0}\, dx = \frac{\pi}{2} [F(x_0 + 0) \qquad - 0)].$$

The integral in (6) can be written as the sum

$$\int_a^{x_0} F(x) \frac{\sin k(x - x_0)}{x - x_0}\, dx + \int_{x_0}^b F(x) \frac{\sin k(x \cdot}{x -}$$

Substituting $x' = x_0 - x$ in the first of these integrals, and $x'' = x - x_0$ in the second, we can write their sum as

$$\int_0^{x}\ \ a \qquad - x')\,\frac{\sin kx'}{x'}\,dx' + \int_0^{b-x_0} F(x'' + x_0)\,\frac{\sin kx''}{x''}\,dx''$$

Lemma 2 applies to each of the integrals here, and since

$$\lim_{x' \to +0} F(x_0 - x') = F(x_0 - 0),$$

and

$$\lim_{x'' \to +0} F(x'' + x_0) = F(x_0 + 0),$$

the limit of their sum is $[F(x_0 - 0) + F(x_0 + 0)]\pi/2$. Statement (6) in the lemma is therefore true.

33. A Fourier Theorem. A theorem which gives conditions under which a Fourier series corresponding to a function converges to that function is called a *Fourier theorem.* One such theorem will now be established. The conditions are only sufficient for the representation; necessary and sufficient conditions are not known.

It will be convenient to consider the function as periodic with period 2π.

Theorem 1. *Let $f(x)$ satisfy these conditions: (a) $f(x + 2\pi) = f(x)$ for all values of x, and (b) $f(x)$ is sectionally continuous in the interval $(-\pi, \pi)$. Then the Fourier series*

$$+ \sum^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

(2)
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad (n = 0, 1, 2, \cdots),$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \qquad \sin nx\, dx \qquad (n = 1, 2, \cdots),$$

converges to the value

at every point where $f(x)$ has a right- and left-hand derivative.

Condition (b) ensures the existence of the Fourier coefficients defined by equations (2), since the products $f(x) \cos nx$ and $f(x) \sin nx$ are continuous by segments and therefore integrable.

It was pointed out in Sec. 26 that series (1) with coefficients (2) can be put in the form

$$\frac{1}{2\pi}\int_{-} \qquad \cos$$

The sum $S_n(x)$ of the first $n + 1$ terms of the series can therefore be written

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \qquad \cos$$

Applying Lagrange's trigonometric identity (Sec. 16) to the sum of cosines here, we have

$$f(x')\,\frac{\text{[illegible]}}{2\sin\left[\frac{1}{2}(x'-x)\right]}\,dx'.$$

The integrand here is a periodic function of x' with period 2π; hence its integral over every interval of length 2π is the same. Let us integrate over the interval $(a, a + 2\pi)$, where the number a has been selected so that the point x is in the interior of that interval; that is, $a < x < a + 2\pi$.

Introducing the factor $(x' - x)$ in both the numerator and the denominator of the integrand, we have

$$(3) \qquad S_n(x) = \frac{1}{\pi} \qquad \frac{\qquad}{x' - x}$$

where

$$F(x') = f(x')\,\frac{x' - x}{2\sin\left[\frac{1}{2}(x'-x)\right]}$$

Now

$$- x$$

Moreover, $F(x')$ is written as the product of two functions each of which is sectionally continuous in every interval and has a derivative from the right and left at the point $x' = x$. This was assumed in the theorem for the first factor $f(x')$, and it is easily verified for the second. Therefore, $F(x')$ is sectionally continuous, and, according to Sec. 31, its derivatives from the right and left exist at $x' = x$.

Therefore $F(x')$ satisfies the conditions of Lemma 3 in which $x_0 = x$ and $k = n + \frac{1}{2}$. Applying that lemma to the integral in equation (3), we have

$$\lim_{n\to\infty} S_n(x) =$$

But according to equation (4),

$$F(x+0) = f(x+0), \qquad F(x-0) = f(x-0),$$

and therefore

$$\lim S_n(x) =$$

This is the same as the statement in the theorem.

34. Discussion of the Theorem. At any point where the periodic function $f(x)$ is continuous,

$$f(x+0) = f(x-0) = f(x);$$

hence at such a point the mean value of the limits of the function, from the right and left, is the value of the function. If the one-sided derivatives of $f(x)$ exist there, the Fourier series converges to $f(x)$.

Suppose $f(x)$ is defined only in the interval $(-\pi, \pi)$. Then it is the periodic extension of this function which is referred to in Theorem 1. Consequently, if $f(x)$ is sectionally continuous, its Fourier series converges to the value

at each interior point where both one-sided derivatives exist. But *at both the end points* $x = \pm\pi$ *the series converges to the value*

$$\tfrac{1}{2}[f(\pi - 0) + f(-\pi + 0)],$$

provided $f(x)$ has a right-hand derivative at $x = -\pi$ and a left-hand derivative at $x = \pi$, because that is the mean value of the periodic function at those points.

It follows that if the series is to converge to $f(-\pi + 0)$ when $x = -\pi$, or to $f(\pi - 0)$ when $x = \pi$, it is necessary that the function have equal limiting values at the end points of its interval; that is,

$$f(-\pi + 0) = f(\pi - 0).$$

It was pointed out that the other forms of Fourier series (Sec. 30) arise from the form used in Theorem 1 by changing the unit or the origin of the variable x. The sine series and cosine series are special cases arising when $f(x)$ is an odd or an

even function. Consequently the Fourier theorem applies to these series at once with the quite obvious modifications necessary because of the changes in the interval.

For the series corresponding to the interval $(-L, L)$, for example, the theorem becomes

Corollary 1. *Let $f(x + 2L) = f(x)$ for all x, and let $f(x)$ be sectionally continuous in the interval $(-L, L)$. Then at any point where $f(x)$ has a right- and left-hand derivative, it is true that*

(1) $\frac{1}{2}|$

$$= \frac{1}{2} a_0 + \sum_{1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_n = \frac{}{L} \int_{-L} \quad \cos \frac{n\pi x}{T} dx \qquad = 0, 1, 2, \cdots),$$

$$\int_{-L}^{L} \quad \sin \frac{n\pi x}{I} dx \qquad (n = 1, 2,$$

It should be observed here, as well as in Theorem 1, that the *existence of the one-sided derivatives is not required at all points* of the interval, but only at those points where representation (1) is used. The function $\sqrt[3]{x^2}$ in the interval $(-L, L)$, for instance, does not have one-sided derivatives at $x = 0$. But, according to our expansion theorem, the Fourier series corresponding to this function must converge to $\sqrt[3]{x^2}$ at all points for which $-L \leqq x < 0$ or $0 < x \leqq L$. At $x = 0$ the convergence is not ensured by our theorem.

Again, if $f(x)$ is defined in the interval $(0, L)$ and is sectionally continuous there, its *Fourier sine series*

(2) $$| b_n \sin \frac{n\pi x}{I},$$

where

$$\frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx \qquad (n = 1, 2,$$

converges to $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at each point x $(0 < x < L)$ where $f(x)$ has one-sided derivatives. Series (2) obviously always converges to zero when $x = 0$ and when $x = L$.

Under the same conditions $f(x)$ is represented by its *Fourier cosine series* in the interval $(0, L)$:

$$(3)\quad \frac{1}{2}[\qquad 0) \qquad - 0)] = \frac{1}{2}\qquad (0 < x < L),$$

where

$$= \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}$$

But in view of the even periodic function represented by the cosine series, this series converges to $f(+0)$ at the point $x = 0$ when the derivative from the right exists at that point. It converges to $f(L - 0)$ at the point $x = L$ when $f(x)$ has a left-hand derivative at that point.

Broader conditions than these, under which the Fourier series converges to its function, will be stated in the next chapter.

35. The Orthonormal Trigonometric Functions. Let us denote by A_L the aggregate or space of all functions defined in the interval $(-L, L)$ which are sectionally continuous there and which possess right- and left-hand derivatives at all points, except the end points, of the interval. At the end points let the derivatives from the interior exist. Also let every function of the class A_L be defined at each point x of discontinuity to have the value $\frac{1}{2}[f(x + 0) + f(x - 0)]$, and at the end points $x = \pm L$ to have the value $\frac{1}{2}[f(L - 0) + f(-L + 0)]$.

Then, according to Corollary 1, for every function $f(x)$ belonging to the class A_L there is a series (the Fourier series) of the functions $\sin (n\pi x/L)$, $\cos (n\pi x/L)$ which converges in the ordinary sense to $f(x)$. This can be stated as follows in the terminology of Chap. III.

Corollary 2. *In the function space A_L, the orthonormal set consisting of all the functions*

$$1 \qquad\qquad -, \frac{\ }{\sqrt{L}} \sin \frac{n\pi x}{\ } \qquad = 1, 2,$$

is closed with respect to ordinary convergence. It is also complete.

The proof of completeness is left for the problems.

Similar statements can be made for functions defined in the interval $(0, L)$, with respect to either the set of sine functions or the set of cosine functions.

Note that the last corollary is a statement about functions whose one-sided derivatives exist at all points of the interval, a condition which is not used in Corollary 1.

Let us observe finally how the conditions of our Fourier theorem apply to our examples. The function in the example treated in Sec. 27; namely,

$$\begin{aligned} f(x) &= 0 \qquad \text{when } -\pi < x \leqq 0, \\ &= x \qquad \text{when } 0 \leqq x < \pi, \end{aligned}$$

is continuous in the interval $(-\pi, \pi)$. It has one-sided derivatives at all points. Series (1), Sec. 27, therefore converges to $f(x)$ at all points in the interval $-\pi < x < \pi$, according to Theorem 1. At the points $x = \pm\pi$ it converges to the value $\pi/2$, since $f(-\pi + 0) = 0$ and $f(\pi - 0) = \pi$. The graph of the periodic function shown there (Fig. 6) would be a complete representation of the function represented by the series if the points $(\pm\pi, \pi/2)$, $(\pm 3\pi, \pi/2)$, $\cdots$ were inserted.

In Sec. 29 the cosine and sine series were found for the function

$$\begin{aligned} & \qquad \text{when } 0 \leqq x < \frac{\pi}{2}, \\ & 0 \qquad \text{when } \frac{\pi}{2} < x \leqq \pi. \end{aligned}$$

This function is sectionally continuous in the interval $(0, \pi)$, and its one-sided derivatives exist there. The sine series therefore converges to $f(x)$ when $0 \leqq x \leqq \pi$, except at $x = \pi/2$, where it converges to $\pi/4$. At $x = 0$ and $x = \pi$ it converges to $f(+0)$ and $f(\pi - 0)$, since these are both zero. The cosine series for this function converges in just the same manner in the interval $(0, \pi)$.

PROBLEMS

1. Show that each of the functions described in Probs. 1 to 4, Sec. 27, satisfies the conditions under which the series found there converges to the function, except possibly at certain points. What is the sum of the series at those points?

Ans. Prob. 1: $x = \pm\pi$; sum $= 0$;
Prob. 2: $x = \pm\pi$; sum $= \cosh\pi$;
Prob. 3: $x = 0, \pm\pi$; sum $= \frac{3}{2}$.

2. Solve Prob. 1 above for each of the functions in Probs. 1 to 7, Sec. 29.

3. Solve Prob. 1 above for each of the functions in Probs. 3 to 9, Sec. 30.

4. If $f(x) = 0$ when $-1 < x < 0$, $f(x) = \cos \pi x$ when $0 < x < 1$, $f(0) = \frac{1}{2}$, $f(1) = -\frac{1}{2}$, and $f(x + 2) = f(x)$ for all x, show that

$$f(x) = \frac{1}{2} \qquad \frac{n}{4n^2 -} \quad \sin$$

for all values of x.

5. If $f(x) = c/4 - x$ when $0 \leqq x \leqq c/2$, $f(x) = x - 3c/4$ when $c/2 \leqq x \leqq c$, show that

$$2c \qquad\qquad \frac{-\,2)\pi x}{}$$

for all x in the interval $0 \leqq x \leqq c$.

6. If $f(x) = x^2$ when $-1 < x \leqq 0$, $f(x) = 0$ when $0 \leqq x < 1$, $f(1) = \frac{1}{2}$, and $f(x + 2) = f(x)$ for all x, find its Fourier series and show that it converges to $f(x)$ for all values of x.

7. Prove that the orthonormal set of functions in Corollary 2 is complete in the function space A_L. (Compare Sec. 22; show that any function in A_L which is orthogonal to every member of the set must be identically zero.)

8. State and prove the corollary, corresponding to Corollary 2, for functions defined in the interval $(0, L)$, with respect to the orthonormal set of functions $\{\sqrt{2/L} \sin (n\pi x/L)\}$.

9. Show that the series

of squares of the coefficients in the Fourier sine series converges whenever $f(x)$ is bounded and integrable on the interval $(0, L)$, and that

$$\overline{T,} \;\; | \;\; [f(x)]^2\, dx.$$

[See formula (5), Sec. 21.]

10. Show that the series

$$\frac{1}{2}\,(\qquad \sum_{1} a_n^2 \qquad \left(a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx\right),$$

involving the squares of the coefficients in the Fourier cosine series,

converges whenever $f(x)$ is bounded and integrable in the interval $(0, L)$, and that

$$\leq \frac{2}{L}$$

(Compare Prob. 9.)

11. If $f(x)$ is bounded and integrable in the interval $(-L, L)$, show that the series

$$\tfrac{1}{2}a_0^2 + \sum^{\infty} a_n^2 + \sum^{\infty} b_n^2,$$

where a_n and b_n are the coefficients in Fourier series (1), Sec. 34, converges to a sum not greater than

(Compare Prob. 9.)

12. For every function which is bounded and integrable in the interval $(-L, L)$, the Fourier coefficients a_n and b_n in series (1), Sec. 34, approach zero as n tends to infinity. Show how this follows from Prob. 11. When the function is sectionally continuous, show that the result for b_n follows also from Lemma 1.

13. The coefficients a_n and b_n, in Corollary 1, are those for which the sum of any fixed finite number of terms of the series written there will be the best approximation in the mean to $f(x)$, in the interval $(-L, L)$. Show how this follows as a special case of Theorem 1, Chap. III.

14. Find the values of A_1, A_2, and A_3 such that the function

$$y \qquad . \sin\frac{\pi x}{2} + A_2 \sin\frac{2\pi x}{2} + A_3 \sin\frac{3\pi x}{2}$$

will be the best approximation in the mean to the function $f(x) = 1$, over the interval $(0, 2)$ (compare Prob. 13). Also draw the graph of y, using the coefficients found, and compare it to the graph of $f(x)$.

Ans. $A_1 = 4/\pi$, $A_2 = 0$, $A_3 = 4/(3\pi)$.

15. Show that it follows from the expansion in Prob. 5, Sec. 30, by setting $x = L$, that

$$\overline{6}.$$

Similarly, show that

$$\frac{\pi^2}{12}.$$

CHAPTER V

FURTHER PROPERTIES OF FOURIER SERIES; FOURIER INTEGRALS

36. Differentiation of Fourier Series. We have seen that the Fourier series representation of the function $f(x) = x$ is valid in the interval $-\pi < x < \pi$; thus (Prob. 1, Sec. 29)

$$x = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \cdots)$$

when $-\pi < x < \pi$. But the series obtained by differentiating this series term by term, namely,

$$2(\cos x - \cos 2x + \cos 3x - \cdots),$$

does not converge to the derivative of x in the interval $(-\pi, \pi)$. The term $\cos nx$ does not approach zero as n tends to infinity; hence the series does not converge.

For all values of x, the above series for the function $f(x) = x$ represents a periodic function with discontinuities at the points $x = \pm\pi, \pm 3\pi, \cdots$. We shall see that the continuity of the periodic function is an important condition for the termwise differentiation of a Fourier series. A complete set of sufficient conditions can be stated as follows:

Theorem 1. *Let $f(x)$ be a continuous function in the interval $-\pi \leqq x \leqq \pi$ such that $f(\pi) = f(-\pi)$, and let its derivative $f'(x)$ be sectionally continuous in that interval. Then the one-sided derivatives of $f(x)$ exist* (Sec. 31), *and hence $f(x)$ is represented by its Fourier series*

$$(1)\quad f(x) = \tfrac{1}{2}a_0 + \sum_{1} (a_n \cos nx + b_n \sin nx) \qquad x \leqq \pi),$$

where

$$(2)\quad a_n = \frac{1}{\pi}\int \quad f(x)\cos \qquad b_n = \frac{1}{\quad} \qquad \sin$$

and at each point where $f'(x)$ has a derivative that series can be differentiated termwise; that is,

$$(3) \qquad (-a_n \sin nx \qquad b_n \cos \qquad (-\pi < x$$

Since $f'(x)$ satisfies the conditions of our Fourier theorem, it is represented by its Fourier series at each point where its derivative $f''(x)$ exists. At such a point $f'(x)$ is continuous, so that

$$(4) \qquad f'(x) = \tfrac{1}{2}a_0' + \sum_{1}^{\infty} (a_n' \cos nx + b_n' \sin nx),$$

where

$$(5) \quad a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \cos nx \, dx, \qquad b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \sin nx \, dx.$$

These integrals can be integrated by parts, since $f(x)$ is continuous and $f'(x)$ is sectionally continuous. Therefore

$$a_n' = \frac{1}{\pi}\Big[f(x) \cos nx\Big]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} \qquad \sin nx \, dx$$

$$(6) \qquad \cos n\pi \; [f(\pi) - f(-\pi)] + nb$$

This reduces to nb_n because of our condition that $f(\pi) = f(-\pi)$. Furthermore, $a_0' = 0$. Likewise,

$$b_n' = \frac{1}{\pi}\Big[f(x) \sin nx\Big]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

Substituting these values of a_n' and b_n' into equation (4), we have

$$f'(x) = \sum^{\infty} (nb_n \cos nx - \qquad \sin$$

This is the equation (3) which was obtained by differentiating (1) term by term; hence the theorem is proved.

It is important to observe that, according to equation (6), *the Fourier series for $f'(x)$ does not reduce to series* (3) *obtained by termwise differentiation if the function fails to satisfy the condition*

This condition ensures the continuity of the periodic extension of $f(x)$ at the points $x = \pm\pi$, and therefore at all points, in view of the continuity of $f(x)$ in the interval $(-\pi, \pi)$.

At a point where $f'(x)$ has a derivative from the right and from the left, but no ordinary derivative, we can easily see from the above proof that termwise differentiation is still valid in the sense that

$$\tfrac{1}{2}[f'(x+0)+f'(x-0)] = \sum_{1}^{\infty} n(-a_n \sin nx + b_n \cos nx).$$

Since this is true for the periodic extension of $f(x)$, the derived series converges at the points $x = \pm\pi$ to the value

if $f'(x)$ has a right-hand derivative at $-\pi$ and a left-hand derivative at π. We are assuming the continuity of the periodic extension of $f(x)$ at all points, of course.

Theorem 1 applies with the usual changes to the other forms of Fourier series.

PROBLEMS

1. Show that the series in Prob. 4, Sec. 27, can be differentiated term by term, and state what function is represented by the derived series. (Compare Prob. 4, Sec. 35.)

2. In the problems, Sec. 29, obtain the series in Prob. 3*a* by differentiating the series in Prob. 2*b*. Note that this is permissible according to Theorem 1; but we cannot reverse the process and obtain the latter series by differentiating the former.

3. In Probs. 1 to 7, Sec. 29, which of the series can be differentiated termwise? *Ans.* 1(*b*); 2(*a*), (*b*); 3(*b*); 4(*b*); 6(*a*), (*b*); 7(*b*).

4. Show that in Probs. 4 and 5, Sec. 30, the series are termwise differentiable.

5. Show that the Fourier coefficients a_n and b_n for the function $f(x)$, described in the first sentence of Theorem 1, satisfy the relations

$$\lim_{n\to\infty} na_n = 0, \qquad \lim_{n\to\infty} nb_n = 0.$$

37. Integration of Fourier Series. Termwise integration of a Fourier series is possible under much more general conditions than those for differentiation. This is to be expected, because an integration introduces a factor n in the denominator of the general term. It will be shown in the following theorem that it is not even essential that the original series converge to its function, in order that the integrated series converge to the

integral of the function. Of course, the integrated series is not a Fourier series if $a_0 \neq 0$, for it contains a term $a_0x/2$.

Theorem 2. *Let $f(x)$ be sectionally continuous in the interval $(-\pi, \pi)$. Then whether the Fourier series corresponding to $f(x)$,*

$$(1) \qquad f(x) \sim \tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

converges or not, the following equality is true:

(2)

$$\int_{-\pi} \qquad \qquad \frac{1}{n} [a_n \sin nx - b_n (\cos nx - \cos n\pi)],$$

when $-\pi \leqq x \leqq \pi$. The latter series is obtained by integrating the former one term by term.

Since $f(x)$ is sectionally continuous, the function $F(x)$, where

$$(3) \qquad F(x) = \int_{-\pi}^{x} f(x)\,dx - \tfrac{1}{2}a_0x,$$

is continuous; moreover

$$F'(x) = f(x) - \tfrac{1}{2}a_0,$$

except at points where $f(x)$ is discontinuous, and even there $F(x)$ has right- and left-hand derivatives. Also,

$$F(\pi) = \int_{-\pi}^{\pi} f(x)\,dx - \tfrac{1}{2}a_0\pi = a_0\pi - \tfrac{1}{2}a_0\pi = \tfrac{1}{2}a_0\pi,$$

and $F(-\pi) = \frac{1}{2}a_0\pi$; hence $F(\pi) = F(-\pi)$. According to our Fourier theorem then, for all x in the interval $-\pi \leqq x \leqq \pi$, it is true that

$$\quad {}_n \cos nx \qquad {}_n \sin$$

where

$$\cos nx\,dx, \qquad B_n = \frac{1}{\pi} \int \quad F(x) \sin$$

Since $F(x)$ is continuous and $F'(x)$ is sectionally continuous, the integrals for A_n and B_n can be integrated by parts. Thus if $n \neq 0$,

1

Similarly, $B_n = a_n/n$; hence

$$(4) \qquad F(x) = \frac{1}{2} A_0 + \sum_1^\infty \frac{1}{n} (a_n \sin nx - b_n \cos nx).$$

But since $F(\pi) = \frac{1}{2} a_0 \pi$,

$$\frac{1}{2} a_0 \pi = \frac{1}{2} A_0 - \sum_1^\infty \frac{1}{n} b_n \cos n\pi.$$

Substituting the value of A_0 given here in equation (4),

$$\sum_1^\infty \frac{1}{n} [a_n \sin nx - b_n(\cos nx - \cos n\pi)].$$

In view of equation (3), equation (2) follows at once.

The theorem can be written for the integral from x_0 to x, when $-\pi \leqq x_0 \leqq \pi$ and $-\pi \leqq x \leqq \pi$, by noting that

$$\int_{x_0}^x f(x)\, dx = \int_{-\pi}^x f(x)\, dx - \int_{-\pi}^{x_0} f(x)\, dx.$$

The other forms of Fourier series can be integrated termwise under like conditions, of course.

Still more general conditions under which the Fourier series can be integrated term by term will be noted in Sec. 39.

PROBLEMS

1. By integrating the expansion found in Prob. 4, Sec. 27, from $-\pi$ to x, obtain the expansion

$$= \frac{x}{\pi} + \frac{1}{2} - \frac{1}{2} \cos x - \frac{1}{\pi} \sum^\infty \frac{1}{n} \frac{\sin 2nx}{4n^2}$$

where $-\pi \leqq x \leqq \pi$, and $F(x) = 0$ when $-\pi \leqq x \leqq 0$, $F(x) = 1 - \cos x$ when $0 \leqq x \leqq \pi$.

2. Integrate the series obtained in Probs. 1 and 3, Sec. 27, from 0 to x, and describe the functions represented by the new series.

38. Uniform Convergence. If A_n and B_n $(n = 1, 2, \cdots, m)$ represent any real numbers, the equation

cannot have distinct real roots. In fact, if it has a real root $x = x_0$, then $A_n x_0 + B_n = 0$ for all n, and the ratio B_n/A_n must be independent of n. The discriminant of the quadratic equation in x is therefore negative or zero; that is,

$$\left(\sum_1^m A_n B_n\right)^2 \leqq \sum_1^m A_n^2 \sum_1^m B_n^2. \tag{1}$$

With the help of this relation, known as *Cauchy's inequality*, we can readily show that the convergence of the Fourier series to the function $f(x)$ described in Theorem 1 is absolute and uniform.

Broader conditions for uniform convergence will be cited in the next section. But it should be noted that a Fourier series cannot converge uniformly in any interval containing a discontinuity of its function, since a uniformly convergent series of continuous functions always converges to a continuous function.

Theorem 3. *Let $f(x)$ be a continuous function in the interval $-\pi \leqq x \leqq \pi$ such that $f(\pi) = f(-\pi)$, and let its derivative $f'(x)$ be sectionally continuous in that interval. Then the Fourier series for the function $f(x)$ converges absolutely and uniformly in the interval $(-\pi, \pi)$.*

The theorem will be proved if we can show that for each positive number ϵ an integer m_0, independent of x, can be found such that

$$\cos nx + b_n \sin nx| < \epsilon$$

when $m > m_0$, for all $m' > m$. The term between the absolute value signs represents, of course, the general term in the Fourier series corresponding to $f(x)$. Since it can be written as

$$\left(\theta = \arctan \frac{b_n}{a_n}\right),$$

it is clear that

$$|a_n \cos nx + b_n \sin nx| \leqq \sqrt{a_n^2 + b_n^2};$$

so it will suffice to show that

(2) 0, $m' > m$).

In the proof of Theorem 1 we found that

(3) $$a_n' = nb_n, \qquad b_n' = -na_n,$$

where a_n' and b_n' are the Fourier coefficients of the function $f'(x)$. Therefore

$$\sum_m^{m'} \sqrt{a_n^2 + b_n^2} = \sum_m^{m'} \frac{1}{n} \sqrt{(a_n')^2 + (b_n')^2}.$$

Applying inequality (1) to the last sum, we have

$$\sum_m^{m'} \sqrt{a_n^2 + b_n^2} \leqq \left\{ \sum_m^{m'} \frac{1}{n^2} \sum_m^{m'} [(a_n')^2 + (b_n')^2] \right\}^{\frac{1}{2}}$$

Bessel's inequality (4), Sec. 21, applies to the bounded integrable function $f'(x)$, with respect to the orthonormal set of functions

(5) $$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\} \quad (n = 1, 2, \cdots),$$

giving the relation

$$\sum_1^{m'} [(a_n')^2 + (b_n')^2] \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} [f'(x)]^2 \, dx$$

for every integer m'. Let M denote the member on the right here; then the second sum on the right of inequality (4) does not exceed the number M.

Now the series

converges; so for any positive number ϵ^2/M an integer m_0 can be found such that

$\overline{M}$'

when $m > m_0$, for all $m' > m$; and m_0 is clearly independent of x. For this choice of m_0, the right-hand member of inequality (4) is less than ϵ, so that inequality (2) is established and the theorem is proved.

But in view of inequality (2) we have also shown that, *under the conditions in Theorem 3, the series*

$$\sum_1^\infty \sqrt{a_n^2 + b_n^2}$$

always converges. Consequently each of the following series converges:

$$\sum_1^\infty |a_n|, \quad \sum_1^\infty |b_n|.$$

It is of interest to note that the Parseval relation (3), Sec. 22, applies to the class of functions described in Theorem 3 with respect to the orthonormal set of trigonometric functions (5). This follows by multiplying the Fourier series expansion of $f(x)$ by $f(x)$, thus leaving it still uniformly convergent, and integrating, to obtain

$$f(x)]^2\,dx = \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} f(x)\,dx + \sum_1^\infty \Big[a_n \int_{-\pi}^{\pi} f(x) \cos nx\,dx + b_n \int_{-\pi}^{\pi} f(x) \sin nx\,dx \Big].$$

In view of the definitions of a_n and b_n, this can be written

(6) $$\int_{-\pi}^{\pi} [f(x)]^2\,dx =$$

This is the Parseval relation.

PROBLEM

Show that if a class of functions satisfies the Parseval relation, the orthonormal set is closed with respect to the limit in the mean (Sec. 22). Hence deduce that set (5) is closed in that sense, for the class of all functions satisfying the conditions in Theorem 3.

39. Concerning More General Conditions. The theory of Fourier series developed above will be sufficient for our purposes. Let us note at this point, however, a few of the many more

general results which are known. These will be stated without proof, since our purpose is only to inform the reader of the existence of such theorems. They cannot be stated in their most general form, usually, without introducing Lebesgue integrals in place of the Riemann integrals considered here.

a. Fourier Theorem. Let $f(x)$ denote here a periodic function with period 2π, and let $\int_{-\pi}^{\pi} f(x)\,dx$ exist. If the integral is improper, let it be absolutely convergent. Then the Fourier series corresponding to $f(x)$ converges to the value

at each point x which is interior to an interval in which $f(x)$ is of bounded variation.*

b. Uniform Convergence. If the periodic function $f(x)$ described under (a) is continuous and of bounded variation in some interval (a, b) then its Fourier series converges to $f(x)$ uniformly in any interval interior to (a, b).†

We have noted earlier that the partial sums $S_n(x)$ of a Fourier series cannot approach the function $f(x)$ uniformly over any interval containing a point of discontinuity of $f(x)$. The nature of the deviation of $S_n(x)$ from $f(x)$ in such an interval is known as the *Gibbs phenomenon.*‡

c. Integration. The Parseval relation

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{1}{2}a_0^2 - \qquad {}_{1} \tag{1}$$

is true whenever $f(x)$ is bounded and integrable in the interval $(-\pi, \pi)$.§ That is, the series of squares of the Fourier coefficients of $f(x)$ on the right of equation (1) converges to the number on the left.

Now let α_n and β_n be the Fourier coefficients of a function $\varphi(x)$, bounded and integrable in the interval $(-\pi, \pi)$. Then $(a_n + \alpha_n)$ and $(b_n + \beta_n)$ are the coefficients of the function $(f + \varphi)$, and according to equation (1) we have

* See first the proof in Ref. 2 at the end of this chapter.

† For a proof, see Ref. 2.

‡ See Ref. 1.

§ See first the proof given in Ref. 2.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |$$

$$\frac{1}{2}$$

Likewise

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(\quad dx$$

$$= \frac{1}{2}(a_0 - \quad [(a_n - \alpha_n \quad (b_n -$$

and by adding the last two equations we find that

$$(2) \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \quad 1$$

In form (2) of the Parseval formula suppose that

$$\begin{aligned} \varphi(x) &= g(x) \quad \text{when } -\pi < x < t, \\ &= 0 \qquad\ \ \text{when } t < x < \pi \ (-\pi \leqq t \leqq \pi), \end{aligned}$$

where $g(x)$ is bounded and integrable in the interval $(-\pi, \pi)$. Then

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{t} g(x) \cos nx\, dx, \qquad \beta_n = \frac{1}{\pi}\int_{-\pi}^{t} g(x) \sin nx\, dx,$$

and form (2) becomes

$$(3) \quad \int_{-\pi}^{t} f(x)g(x)\, dx = \tfrac{1}{2}a_0 \int_{-\pi}^{t} g(x)\, dx$$
$$a_n \int_{-\pi}^{t} g(x) \cos nx\, dx + b_n \int_{-\pi}^{t} g(x) \sin nx\, dx \Bigg].$$

So it follows from statement (c) that if the Fourier series corresponding to any bounded integrable function $f(x)$ is multiplied by any other function of the same class and then integrated term by term, the resulting series converges to the integral of the product $f(x)g(x)$. When $g(x) = 1$, we have a general theorem for the termwise integration of a Fourier series.

PROBLEM

Assuming statement (c), show that it follows that the set of functions (5), Sec. 38, is closed, in the sense of convergence in the mean, with respect to the class of bounded integrable functions in the interval $(-\pi, \pi)$. (Compare the problem at the end of Sec. 38.)

40. The Fourier Integral. The Fourier series (Sec. 30) corresponding to $f(x)$ in the interval $(-L, L)$ can be written

$$(1)\quad \frac{1}{2L}\int_{-L}^{L} f(x')\,dx' + \frac{1}{L}\sum_{1}^{\infty}\int_{-L}^{L} f(x')\cos\left[\frac{n\pi}{L}(x'-x)\right]dx'.$$

It converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ when $-L < x < L$, provided $f(x)$ is sectionally continuous and has right- and left-hand derivatives in the interval $(-L, L)$. If $f(x)$ satisfies those conditions in *every* finite interval, then L may be given any fixed value, arbitrarily large but finite, in order that we may obtain a representation of $f(x)$ in a large interval. But this series representation cannot be valid outside that interval unless $f(x)$ is periodic with the period $2L$, since series (1) represents only such functions.

To indicate a representation which may be valid for all real x when $f(x)$ is not periodic, it is natural to try to extend series (1) to the case $L = \infty$. The first term would then vanish, assuming that $\int_{-\infty}^{\infty} f(x)\,dx$ converges. Putting $\Delta\alpha = \pi/L$, the remaining terms can be written

$$\frac{1}{L}\sum_{1}^{\infty}\int_{-L}^{L} f(x')\cos\left[\frac{n\pi}{L}(x'-x)\right]dx'$$
$$= \frac{1}{\pi}\sum_{1}^{\infty}\Delta\alpha\int_{-L}^{L} f(x')\cos\left[n\Delta\alpha(x'-x)\right]dx'.$$

The last series has the form $\sum_{1}^{\infty} F(n\Delta\alpha)\Delta\alpha$, where

$$f(x')\cos\left[\alpha(x'-x)\right]dx';$$

hence when $\Delta\alpha$ is small, it may be expected to approximate the integral $\int_{0}^{\infty} F(\alpha)\,d\alpha$. (Note, however, that its limit as $\Delta\alpha$

approaches zero is not the definition of this integral; furthermore, when $\Delta\alpha$ approaches zero, L becomes infinite, so $F(\alpha)$ itself changes.) But if the process were sound, when L becomes infinite series (1) would become

$$\frac{1}{\pi}\int_0^\infty d\alpha \int_{-}^{\infty} f(x') \cos [\alpha(x' - x)]\, dx'.$$

This is the *Fourier integral* of $f(x)$. Its convergence to $f(x)$ for all finite values is suggested but by no means established by the above argument. It will now be shown that this representation is valid when $f(x)$ satisfies the conditions in the following *Fourier integral theorem:*

Theorem 4. *Let $f(x)$ be sectionally continuous in every finite interval (a, b), and let $\int_{-\infty}^{\infty} |f(x)|dx$ converge. Then at every point x $(-\infty < x < \infty)$, where $f(x)$ has a right- and left-hand derivative, $f(x)$ is represented by its Fourier integral as follows:*

$$(2)\quad \frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} f(x') \cos [\alpha(x' - x)]\, dx'.$$

In every interval (a, b), $f(x)$ satisfies the conditions of Lemma 3, Sec. 32, so that

$$(3)\quad \frac{\pi}{2}[\qquad 0) \qquad - 0)] = \lim_{\to}$$

at any point x $(a < x < b)$, where $f(x)$ has a right- and left-hand derivative. Now

Whenever $a < x$,

$$\int_{-}^{a} f(x') \sin \qquad x' - x \qquad \leqq \int_{-\infty}^{a} \frac{|f(x')|}{|x' - x|}\, dx',$$

and the latter integral converges because $\int_{-\infty}^{\infty} |f(x)|\, dx$ does.

Similarly for the last integral in equation (4), when $b > x$. Hence for any $\epsilon > 0$ a positive number N can be found such that if $a < -N$ and $b > N$, the first and last integrals on the right of equation (4) will each be numerically less than $\epsilon/3$. The second integral there can be made to differ from the value $\frac{1}{2}\pi[f(x + 0) + f(x - 0)]$ by an amount numerically less than $\epsilon/3$ by taking α sufficiently large, according to equation (3). Hence the integral (4) differs numerically from the above value by an amount less than ϵ for all α greater than some fixed number; that is,

$$(5) \quad \lim_{\to \infty} \int_{-\infty}^{\infty} f(x') \qquad x' = \frac{\pi}{2} [f(x + 0) + f(x - 0)].$$

Writing the fraction in the integrand as an integral, and dividing by π, this becomes

$$\frac{1}{2} [f(x + 0$$

$$= \lim_{\alpha \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x')\, dx' \int_0^{\alpha} \cos [\alpha'(x' - x)]\, d\alpha'$$

$$= \lim_{\alpha \to \infty} \frac{1}{\pi} \int_0^{\alpha} d\alpha' \int_{-\infty}^{\infty} f(x') \cos [\alpha'(x' - x)]\, dx'.$$

The inversion of order of integration in the last step is valid because the integrand does not exceed $|f(x')|$ in absolute value, so that the integral

$$\int_{-\infty}^{\infty} f(x') \cos [\alpha'(x' - x)]\, dx'$$

converges uniformly for all α'.* The last equation is the same as statement (2) in the theorem.

Fourier integral theorems with somewhat broader conditions on $f(x)$ are also known. The more modern theorems take advantage of the use of Lebesgue integration.

PROBLEMS

1. Verify the Fourier integral theorem directly for the function $f(x) = 1$ when $-1 < x < 1$, $f(x) = 0$ when $x < -1$ and when $x > 1$. The following integration formula, usually established in advanced calculus, will be useful:

* See, for instance, p. 199 of Ref. 1.

$$\int_0^\infty \frac{\sin kx}{x}\,dx = \frac{\pi}{2} \qquad \text{if } k > 0,$$
$$= 0 \qquad \text{if } k = 0,$$
$$= \frac{-\pi}{2} \qquad \text{if } k < 0.$$

2. Show that the function $f(x) = 0$ when $x < 0$, $f(x) = e^{-x}$ when $x > 0$, $f(0) = \frac{1}{2}$, is represented by its Fourier integral; hence show that the integral

$$\int_0^\infty \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2}\,d\alpha$$

has the value 0 if $x < 0$, $\pi/2$ if $x = 0$, and πe^{-x} if $x > 0$.

3. Show that the Fourier integral of the function $f(x) = 1$ does not converge.

41. Other Forms of the Fourier Integral. Let $f(x)$ be an *odd* function which satisfies the conditions of Theorem 4. Then

$$\int_{-\infty}^\infty f(x')\cos[\alpha(x' - x)]\,dx'$$
$$= \int_0^\infty f(x')\cos[\alpha(x' - x)]\,dx' + \int_0^\infty f(-y)\cos[\alpha(y + x)]\,dy$$
$$= \int_0^\infty f(x')\cos[\alpha(x' - x)]\,dx' - \int_0^\infty f(x')\cos[\alpha(x' + x)]\,dx'$$
$$= 2\sin\alpha x \int_0^\infty f(x')\sin\alpha x'\,dx'.$$

Hence the Fourier integral formula becomes

$$\text{(1)}\quad \frac{1}{2}[f(x+0) + f(x-0)] = \frac{2}{\pi}\int_0^\infty \sin\alpha x\,d\alpha \int_0^\infty f(x')\sin\alpha x'\,dx'.$$

This is the *Fourier sine integral*, corresponding to the Fourier sine series. If $f(x)$ is defined only when $x > 0$, formula (1) is valid provided $f(x)$ is piecewise continuous in each finite interval in $x \geqq 0$ and has a right- and left-hand derivative at the point x ($x > 0$), and provided $\int_0^\infty |f(x)|\,dx$ converges.

Similarly if $f(x)$ is an *even* function satisfying the conditions of Theorem 4, it is represented by its *Fourier cosine integral:*

(2) $\frac{1}{2}|$

$$= \frac{2}{\pi} \int_0^\infty \cos \alpha x \, d\alpha \int_0^\infty f(x') \cos \alpha x'$$

Under the conditions just given for the sine integral, formula (2) is also valid if $f(x)$ is defined only when $x > 0$. Moreover, the integral converges at $x = 0$ to $f(+0)$ provided $f(x)$ has a right-hand derivative there.

By writing $\cos [\alpha(x' - x)]$ in terms of imaginary exponential functions, the integral formula of Theorem 4 can be reduced to

(3) $\frac{1}{2} [f(x + 0$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\alpha x} \, d\alpha \int_{-}$$

This is the *exponential form* of the Fourier integral of the function $f(x)$ defined for all real values of x.

If $g(\alpha)$ is a known function when $\alpha > 0$, note that the integral equation

(4)

can be solved easily for the unknown function $f(x)$ $(x > 0)$, provided that function is one of the class for which the Fourier integral formula (1) is true. For by multiplying equation (4) through by $\sqrt{2/\pi} \sin \alpha x$ and integrating with respect to α over the interval $(0, \infty)$ we have, in view of formula (1),

$$(5) \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\alpha) \sin \alpha x \, d\alpha \qquad (x > 0).$$

Of course this formula would give the mean value of $f(x)$ at a point of discontinuity.

The integral in equation (4) is called the Fourier sine transform of $f(x)$. Formula (5), which gives $f(x)$ in terms of its transform $g(\alpha)$, has precisely the same form as equation (4).

In view of formula (2), the sine functions can clearly be replaced by cosines in equations (4) and (5).

PROBLEMS

1. Show that the formula in Theorem 4 reduces to formula (2) when $f(x)$ is an even function.

2. Transform the formula in Theorem 4 to the exponential form (3).

3. Apply formula (2) to the function $f(x) = 1$ when $0 \leqq x < 1$, $f(x) = 0$ when $x > 1$, and hence show that

$$\begin{aligned} \int_0 \frac{\sin \alpha \cos \alpha x}{\alpha}\, d\alpha &= \frac{\pi}{2} \qquad \text{when } 0 \leqq x < \\ &= \frac{\pi}{4} \qquad \text{when } x = 1, \\ &= 0 \qquad \text{when } x > 1. \end{aligned}$$

Apply formula (1) to the function $f(x) = e^{-x} \cos x$, and thus show that

$$\int_0 \qquad\qquad \text{if } x > 0.$$

5. By applying formula (1) to the function $f(x) = \sin x$ when $0 \leqq x \leqq \pi$, $f(x) = 0$ when $x > \pi$, show that

$$\begin{aligned} \int_0 \frac{\sin \alpha x \sin \pi\alpha}{1 - \alpha^2}\, d\alpha &= \frac{\pi}{2} \sin x \qquad \text{if } 0 \leqq x \\ &= 0 \qquad \text{if } x > \pi. \end{aligned}$$

6. Apply formula (2) to the function $f(x)$ of Prob. 5 and obtain another integration formula.

7. Show that the solution of the integral equation

$$\int_0^\infty f(x) \sin \alpha x \, dx = g(\alpha),$$

where $g(\alpha) = 1$ when $0 < \alpha < \pi$, $g(\alpha) = 0$ when $\alpha > \pi$, is

$$f(x) = \frac{2}{\pi} \frac{1 - \cos \pi x}{x} \qquad > 0).$$

✓**8.** Show that the integral equation

$$\int_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$$

has the solution

REFERENCES

1. Carslaw, H. S.: "Fourier's Series and Integrals," 1930.
2. Whittaker, E. T., and G. N. Watson: "Modern Analysis," Chap. 9, 1927.
3. Titchmarsh, E. C.: "Theory of Functions," Chap. 13, 1939. This is more advanced.

CHAPTER VI

SOLUTION OF BOUNDARY VALUE PROBLEMS BY THE USE OF FOURIER SERIES AND INTEGRALS

42. Formal and Rigorous Solutions. In an introductory treatment of boundary value problems in the partial differential equations of physics, it seems best to follow to some extent the plan used in introductory courses in ordinary differential equations; that is, to stress the method of obtaining a solution of the problem as stated, and give less attention to the precise statement of the problem that would ensure that the solution found is the only one possible. But it is important that the student be aware of the shortcomings of this sort of treatment; hence some discussion of the rigorous statement and solution of problems will be given. The subject of boundary value problems in partial differential equations is still under development; in particular, the uniqueness of the solutions of some of the important types of problems has not yet been satisfactorily investigated.

In ordinary differential equations, the solution for all $x \geqq 0$ of the simple boundary value problem

$$y'(x) = 2, \qquad y(0) = 0,$$

would generally be given as $y = 2x$, because it is understood that $y(x)$ must be continuous. Without such an agreement, however, the function $y = 2x + c$ when $x > 0$, $y = 0$ when $x = 0$, is a solution for every constant c; that is, the solution is not unique. Even when the boundary condition is written $y(+0) = 0$, the solution could be written, for instance, as $y = 2x$ when $0 \leqq x \leqq a$, $y = 2x + c$ when $x > a$, unless $y(x)$ is required to be continuous for all $x \geqq 0$.

Such tacit agreements necessary for the existence of just one solution are not nearly so evident in partial differential equations. Furthermore, if the result is found only in the form of an infinite series or integral, it is sometimes quite difficult

to determine the precise conditions under which that series or integral converges and represents even one possible solution.

The treatment of an applied boundary value problem is only a *formal* one unless it is shown (*a*) that the result found is actually a solution of the differential equation and satisfies all the boundary conditions, and (*b*) that no other solution is possible. The physical problem will require that there should be only one solution; hence the mathematical statement of the problem is not strictly complete unless the uniqueness condition (*b*) is satisfied.

43. The Vibrating String. The formula for the displacements $y(x, t)$ in a string stretched between the points $(0, 0)$ and $(L, 0)$ and given an initial displacement $y = f(x)$ was found in Sec. 13 to be

$$y = \sum_1^\infty A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}, \tag{1}$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

The function $f(x)$ must of course be continuous in the interval $0 \leqq x \leqq L$ and vanish when $x = 0$ and $x = L$. In addition, let $f(x)$ be required to have a right- and left-hand derivative at each point. Then the Fourier sine series obtained when $t = 0$ in formula (1) does converge to $f(x)$; hence this initial condition is actually satisfied. Thus an important improvement in the formal solution is made possible by the theory of Fourier series.

The nature of the problem requires the solution $y(x, t)$ to be continuous with respect to x and t. Since $y(x, t)$ is to satisfy the equation of motion

$$(t > 0,\ 0 < x < L), \tag{2}$$

and all the boundary conditions

$$y(0, t) = 0, \qquad y(L, t) = 0,$$
$$\frac{\partial y(x, 0)}{\partial t} = 0, \qquad y(x, 0) = f(x),$$

some conditions relative to the existence of its derivatives must also be satisfied. We shall now examine the function

defined by formula (1) to see if it is actually a solution of our problem.

The Solution Established. It is possible to sum the series in formula (1); that is, to write the result in a closed, or finite, form. This will make it much easier to examine the function $y(x, t)$.

Since

$$2 \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} = \sin \left[\frac{n\pi}{L}(x - at)\right] + \sin \left[\frac{n\pi}{L}(x + at)\right],$$

equation (1) can be written

$$(3) \quad y = \frac{1}{2}$$

$$A_n \sin \left[\frac{n\pi}{L}(x + at)\right]$$

The two series here are those obtained by substituting $(x - at)$ and $(x + at)$, respectively, for the variable x in the Fourier sine series for $f(x)$. Since the sine series represents an odd periodic function, the last equation can be written

$$(4) \qquad y = \tfrac{1}{2}[F(x - at) + F(x + at)],$$

where the function $F(x')$ is defined for all real values of x' as the odd periodic extension of $f(x')$; that is,

$$F(x') = f(x') \qquad \text{if}$$

$$F(-x') = -F(x'),$$

and

$$F(x' + 2L) = F(x') \qquad \text{for all } x'.$$

The function $f(x)$ is continuous in the interval $(0, L)$ and vanishes at the end points; hence $F(x')$ is continuous for all x'. According to our Fourier theorem, the two sine series in equation (3) converge to the functions in equation (4) whenever $f(x)$ has one-sided derivatives. The same function $y(x, t)$ is then represented by each of the three formulas (1), (3), and (4); moreover, according to (4), $y(x, t)$ is a continuous function of x and t for all values of these variables.

By differentiating equation (4) we can easily see that $y(x, t)$ satisfies differential equation (2) whenever the derivative $F''(x')$ exists. When it is observed that $F'(x')$ and $F''(x')$ are even and odd functions, respectively, it can be seen that the second derivative exists for all x' provided $f(x)$ has a second derivative whenever $0 < x < L$, and provided that the one-sided derivatives of $f'(x)$ at the end points $x = 0$ and $x = L$ exist and *have the value zero.*

Under these rather severe conditions on $f(x)$, then, our function $y(x, t)$ satisfies the equation of motion for all x and t, and it is also evident from equation (4) that $\partial y/\partial t$ is continuous and vanishes when $t = 0$. The remaining boundary conditions are clearly satisfied, in view of either equation (1) or (4); hence $y(x, t)$ is established as a solution.

If we permit $f'(x)$ and $f''(x)$ to be only sectionally continuous, or if the one-sided second derivatives of $f(x)$ do not vanish at the points $x = 0$ and $x = L$, then at each instant t there will be a finite number of points x at which the second derivatives of $y(x, t)$ fail to exist. Except at these points, differential equation (2) will still be satisfied. In this case we have a solution of our problem in a broader sense.

In either case an examination of the uniqueness of the solution found would be necessary to make the treatment of the problem complete.

An Approximate Solution. Except for the nonhomogeneous boundary condition

(5) $$y(x, 0) = f(x),$$

our boundary value problem is satisfied by the sum of any finite number of terms of the series in equation (1), say

(6) $$y_N = \quad \ldots \; n\pi x \; \ldots \; n\pi at$$

where N is some integer. In place of condition (5) this function satisfies the condition

(7) $$x, 0) = \quad \overset{N}{} \quad \sin$$

The function $y_N(x, t)$ has continuous derivatives of all orders.

The sum in condition (7) is that of the first N terms of the Fourier sine series for $f(x)$. According to Theorem 3, Chap. V, that series converges uniformly to $f(x)$ provided $f'(x)$ is continuous by segments. Hence, by taking N sufficiently large, the sum can be made to approximate $f(x)$ arbitrarily closely for all values of x in the interval $0 \leqq x \leqq L$.

The function $y_N(x, t)$ is therefore established as a solution of the "approximating problem," obtained by replacing condition (5) of the original boundary problem by condition (7).

Similar approximations can be made to the problems to be considered later on. But the remarkable feature in the present case is that the approximating function $y_N(x, t)$ does not deviate from the actual displacement $y(x, t)$ by more than the maximum deviation of $y_N(x, 0)$ from $f(x)$. This is true because $y_N(x, t)$ can be written

$$\frac{n\pi}{L}(x - at)\Big] + \sum^{N} A_n \sin$$

and each sum here consists of the first N terms of the sine series for the odd periodic extension of $f(x)$, except for substitutions of new variables. But the greatest deviation of the first sum from $F(x - at)$, or of the second from $F(x + at)$, is the same as the greatest deviation of $y_N(x, 0)$ from $f(x)$.

PROBLEMS

1. Show that the motion of every point of the string in the above problem is periodic in t with the period $2L/a$.

2. The position of the string at any time t can be found by moving the curve $y = \frac{1}{2}F(x)$ to the right with the velocity a and an identical curve to the left at the same rate and adding the ordinates, in the interval $0 \leqq x \leqq L$, of the two curves so obtained at the instant t. Show how this follows from formula (4).

3. Plot a few positions of the plucked string of Sec. 14, using the method of Prob. 2 above.

44. Variations of the Problem. If each point of the string is given an initial velocity in its position of equilibrium, the boundary value problem in the displacement $y(x, t)$ is the following:

$$y(0, t) = 0, \qquad y(L, t) = 0,$$
$$y(x, 0) = 0,$$

As before, functions of the form $y = X(x)T(t)$ which satisfy the differential equation and all the homogeneous boundary conditions can be found. Writing the series of these particular solutions, we have

$$y = \quad {}^{\infty} \quad A_n \sin \frac{n\pi x}{T} \sin \frac{n\pi at}{}$$

The final condition, that $\partial y/\partial t = g(x)$ when $t = 0$, shows that the numbers $n\pi aA_n/L$ should be the Fourier sine coefficients of $g(x)$; hence the solution of the problem becomes

$$(1) \qquad y = \frac{2}{\pi a} \sum_1^\infty \frac{1}{n} \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \int_0^L g(x') \sin \frac{n\pi x'}{L}\, dx'.$$

By the method of the last section, $\partial y/\partial t$ can be written here in terms of the odd periodic extension $G(x')$ of the function $g(x')$. This leads to the closed forms

$$(2) \qquad y = \frac{1}{2} \int_0^t [G(x - \qquad + G(x + at')]\, dt'$$
$$= \frac{1}{2a} \int_{x-at}^{x+at} G($$

of solution (1). The details of these derivations are left for the problems.

Superposition of Solutions. If the string is given both an initial displacement and initial velocity, the last two boundary conditions become

$$(3) \qquad y(x, 0) = f(x),$$

All the other conditions of the linear boundary problem are homogeneous. They are satisfied by the solution of the problem of the preceding section and by solution (2) above, and therefore by the sum of those two functions, namely

$$(4) \qquad \frac{1}{2a} \int_{x-at}^{x+at} ($$

When $t = 0$, the first function of the sum becomes $f(x)$ and the second vanishes; hence the first of conditions (3) is satisfied. Likewise it is seen that the second of those conditions is satisfied, and therefore equation (4) is the required solution.

In general the solution of a linear problem containing more than one nonhomogeneous boundary condition can be written as the sum of solutions of problems each of which contains only one nonhomogeneous condition. Of course we cannot always find the solutions of the simpler problems which are to be superposed in this way.

Units. It is often possible and advantageous to select units so that some of the constants in our problem become unity. For example, if we write τ for the product (at), the equation of motion of the string reduces to

$$\frac{\partial^2 y}{\partial \tau^2} = \frac{\partial^2 y}{\partial x^2}.$$

Such changes sometimes help to bring out reductions in computation, or general properties of the solution.

Since the boundary problem of the last section, for example, does not involve the number a when the problem is written in terms of τ and x, its solution must be a function only of x, L, and the product (at). This conclusion is possible without our knowing the formula for the solution. But a^2 is proportional to the tension in the string; hence if $y_1(x, t)$ and $y_2(x, t)$ are the displacements when the tension has the values P_1 and P_2, respectively, then

$$y_1(x, t_1) = y_2(x, t_2) \quad \text{if} \tag{5}$$

That is, the same set of instantaneous positions is assumed by the string whether the tension is P_1 or P_2, but the times t_1 and t_2 required to reach any one position are in the ratio $\sqrt{P_2/P_1}$.

Nonhomogeneous Differential Equations. The substitution of a new unknown function sometimes reduces a linear differential equation which is not homogeneous to one which is homogeneous, so that our method of solution can be employed.

To illustrate this, consider the problem of displacements in a stretched string upon which an external force acts proportional to the distance from one end. If the initial displacement and velocity are zero, the units for t and x can be so selected that the

problem becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + Ax \qquad (0 < x < 1, t > 0),$$

$$y(0, t) = 0, \qquad y(1, t) = 0,$$

$$y(x, 0) = 0, \qquad \frac{\partial y(x, 0)}{\partial t} = 0.$$

In terms of the new function Y, where

$$y(x, t) = Y(x, t) + \psi(x),$$

and $\psi(x)$ is to be determined later, the differential equation becomes

$$\frac{\partial^2 Y}{\partial t^2} \qquad \partial^2 Y \qquad Ax \qquad (0 < x < 1, t > 0).$$

This will be homogeneous if

(6) $$\psi''(x) = -Ax \qquad (0 < x < 1).$$

The first pair of boundary conditions on Y are

$$Y(0, t) + \psi(0) = 0, \qquad Y(1, t) + \psi(1) = 0;$$

hence these are homogeneous if

(7) $$\psi(0) = 0, \qquad \psi(1) = 0.$$

In view of conditions (6) and (7),

(8) $$A \qquad (0 < x < 1),$$

and with this choice of ψ the problem in Y becomes a special case of the problem in the preceding section; for the initial conditions are

$$x, 0) = -\psi \qquad \frac{\partial Y(x, 0)}{\partial t} \quad 0.$$

The solution of our problem in forced vibrations therefore can be written

(9) $$y = \qquad - t) + \qquad + t)],$$

where $\Psi(x')$ is the odd periodic extension of the function $\psi(x')$ defined by equation (8) in the interval (0, 1).

PROBLEMS*

1. Carry out the details in the derivation of formula (1).

2. Write out the steps used in deriving formulas (2) from (1).

3. Show that relation (5) fails to hold between the displacements of a given string under different tensions if the initial velocity is the same in both cases and not zero. What change in initial velocity must accompany an increase in tension to cause a more rapid vibration with the same amplitude?

4. A string is stretched between the points (0, 0) and (1, 0). If it is initially at rest on the x-axis, find its displacements under a constant external force proportional to $\sin \pi x$ at each point. Verify your solution by showing that it satisfies the equation of motion and all boundary conditions. *Ans.* $y = A/(\pi^2 a^2) \sin \pi x(1 - \cos \pi a t)$.

5. A wire stretched between two fixed points of a horizontal line is released from rest while it lies on that line, its subsequent motion being due to the force of gravity and the tension in the wire. Set up and solve the boundary value problem for its displacements. Show that its solution can be written in the form (9), if $a = 1$ and $\psi(x) = (x^2 - Lx)g/2$ in the interval $(0, L)$, where g is the acceleration of gravity.

45. Temperatures in a Slab with Faces at Temperature Zero. Let a slab of homogeneous material bounded by the planes $x = 0$ and $x = \pi$ have an initial temperature $u = f(x)$, varying only with the distance from the faces, and let its two faces be kept at temperature zero. The formula for the temperature u at every instant and at all points of the slab is to be determined.

In this problem it is clear that the temperature is a function of the variables x and t only; hence at each interior point this function $u(x, t)$ must satisfy the heat equation for one-dimensional flow,

$$\frac{\quad}{\partial t} \qquad < x < \pi,\ t > 0). \tag{1}$$

In addition, it must satisfy the boundary conditions

$$u(+0, t) = 0, \qquad u(\pi - 0, t) = 0 \qquad (t > 0), \tag{2}$$

$$u(x, +0) = f(x) \qquad (0 < x < \pi). \tag{3}$$

The boundary value problem (1)-(3) is also the problem of temperatures in a right prism or cylinder whose length is π

* Only formal solutions of the boundary value problems here and in the sets of problems to follow are expected, unless it is expressly stated that the solution is to be completely established.

(taken so for convenience in the computation), provided its lateral surface is insulated. Its ends $x = 0$ and $x = \pi$ are held at temperature zero and its initial temperature is $f(x)$.

To find particular solutions of equation (1) that satisfy conditions (2), we write $u = X(x)T(t)$. When substituted in equation (1), this gives $XT' = kX''T$, or

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the function on the left can vary only with x and the one on the right only with t, they must both equal a constant α; that is,

$$X'' - \alpha X = 0, \qquad T' - \alpha k T = 0. \tag{4}$$

Moreover, if the function XT is to satisfy conditions (2), then

$$X(0) = 0, \qquad X(\pi) = 0, \tag{5}$$

provided $X(x)$ is a continuous function.

The solution of the first of differential equations (4) that satisfies the first of conditions (5) is $X = C_1 \sinh x\sqrt{\alpha}$, and this can satisfy the second of conditions (5) only if

$$\alpha = -n^2 \qquad (n = 1, 2,$$

Then $X = C_2 \sin nx$. The solution of the second of equations (4) is, then, $T = C_3 e^{-n^2kt}$. Hence the solutions of equations (1) and (2) of the form $u = XT$ are

$$b_n e^{-n^2kt} \sin nx \qquad (n = 1, 2, \cdots), \tag{6}$$

where the constants b_n are arbitrary.

Clearly no sum of a finite number of functions (6) can satisfy the nonhomogeneous condition (3) unless $f(x)$ happens to be a linear combination of sines of multiples of x. But the infinite series of those functions,

$$u(x, t) = \qquad \sin nx \tag{7}$$

does in general reduce to $f(x)$ in $(0, \pi)$ when $t = 0$, provided the coefficients b_n are those of the Fourier sine series for $f(x)$; namely,

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx \quad (n = 1, 2, \cdots).$$

PROBLEMS*

1. Carry out the details in the derivation of formula (1).

2. Write out the steps used in deriving formulas (2) from (1).

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In this problem it is clear that the temperature is a function of the variables x and t only; hence at each interior point this function $u(x, t)$ must satisfy the heat equation for one-dimensional flow,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi,\ t > 0). \tag{1}$$

In addition, it must satisfy the boundary conditions

$$u(+0, t) = 0, \qquad u(\pi - 0, t) = 0 \qquad (t > 0), \tag{2}$$

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$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the function on the left can vary only with x and the one on the right only with t, they must both equal a constant α; that is,

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$$\alpha = -n^2 \qquad (n = 1, 2, \cdots).$$

Then $X = C_2 \sin nx$. The solution of the second of equations (4) is, then, $T = C_3 e^{-n^2kt}$. Hence the solutions of equations (1) and (2) of the form $u = XT$ are

$$b_n e^{-n^2kt} \sin nx \qquad (n = 1, 2, \cdots), \tag{6}$$

where the constants b_n are arbitrary.

Clearly no sum of a finite number of functions (6) can satisfy the nonhomogeneous condition (3) unless $f(x)$ happens to be a linear combination of sines of multiples of x. But the infinite series of those functions,

$$u(x, t) = \sum_1^\infty b_n e^{-n^2kt} \sin nx, \tag{7}$$

does in general reduce to $f(x)$ in $(0, \pi)$ when $t = 0$, provided the coefficients b_n are those of the Fourier sine series for $f(x)$; namely,

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx \quad (n = 1, 2, \cdots).$$

More precisely, if $f(x)$ is sectionally continuous and has one-sided derivatives at all points in $(0, \pi)$, then

$$u(x, 0) = \sum_1^\infty b_n \sin nx = \tfrac{1}{2}[f(x+0) + f(x-0)] \quad (0 < x < \pi),$$

and this represents $f(x)$ at each interior point where $f(x)$ is continuous.

With those mild restrictions on $f(x)$, then, the solution of the problem is

$$u(x, t) = \frac{2}{\pi} \sum_1^\infty e^{-n^2kt} \sin nx \int_0^\pi f(x') \sin nx' \, dx', \tag{8}$$

provided this series converges to a function $u(x, t)$ such that $u(x, +0) = u(x, 0)$ when $0 < x < \pi$, $u(+0, t) = u(0, t)$ and $u(\pi - 0, t) = u(\pi, t)$ when $t > 0$, and provided the series can be differentiated termwise once with respect to t and twice with respect to x when $t > 0$ and $0 < x < \pi$. It will be shown in the next section that the series does satisfy those conditions.

PROBLEMS

1. Solve the above problem if the faces of the slab are the planes $x = 0$ and $x = L$.

Ans. $u(x, t) = \dfrac{2}{L} \displaystyle\sum_1^\infty \exp\left(-\frac{n^2\pi^2kt}{L^2}\right) \sin \frac{n\pi x}{L} \int_0^L f(x') \sin \frac{n\pi x'}{L} \, dx'.$

2. Find the formula for the temperatures in a slab of width L which is initially at the uniform temperature u_0, if its faces are kept at temperature zero.

$\infty \quad \lceil \quad (2n$

3. The initial temperature in a bar with ends $x = 0$ and $x = \pi$ is $u = \sin x$. If the lateral surface is insulated and the ends are held at zero, find the temperature $u(x, t)$. Verify your result completely. How does the temperature distribution vary with time?

Ans. $u = e^{-kt} \sin x$.

4. Write the solution of Prob. 1 if $f(x) = A$ when $0 < x < L/2$, $f(x) = 0$ when $L/2 < x < L$.

Ans. $u(x, t) = \dfrac{4}{\;}$

5. Two slabs of iron, each 20 cm. thick, one at temperature 100°C. and the other at temperature 0°C. throughout, are placed face to face in perfect contact, and their outer faces are kept at 0°C. (compare Prob. 4). Given that $k = 0.15$ c.g.s. (centimeter-gram-second) unit, find to the nearest degree the temperature 10 min. after contact was made, at a point on their common face and at points 10 cm. from it.

Ans. 37°C.; 33°C.; 19°C.

6. If the slabs in Prob. 5 are made of concrete with $k = 0.005$ c.g.s. unit, how long after contact will it take the points to reach the same temperatures found in the iron slabs after 10 min.? *Ans.* 5 hr.

46. The Above Solution Established. Uniqueness. It is not difficult to show that the series found in Sec. 45, namely

$$\sin nx, \tag{1}$$

represents a function $u(x, t)$ which satisfies all the conditions of the boundary value problem, provided the initial temperature function $f(x)$ is sectionally continuous in the interval $(0, \pi)$ and has one-sided derivatives at all interior points of that interval. For the sake of convenience, we define the value of $f(x)$ to be $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at each point x where the function is discontinuous.

Since $|f(x)|$ is bounded,

$$\int_0^\pi f(x) \sin nx \, dx \right| \leqq \frac{2}{\pi} \int_0^\pi |f(x)| \, dx < M,$$

where M is a fixed number independent of n. Consequently, for each $t_0 > 0$,

$$|b_n e^{-n^2kt} \sin nx| < Me^{-n^2kt_0} \quad \text{when } t \geqq t_0.$$

The series of the constant terms $e^{-n^2kt_0}$ converges; hence, according to the Weierstrass M-test, series (1) converges uniformly with respect to x and t when $t \geqq t_0$, $0 \leqq x \leqq \pi$. Also, the terms of series (1) are continuous with respect to x and t, so that the function $u(x, t)$ represented by the series is continuous for those values of x and t; consequently, whenever $t > 0$,

$$u(+0, t) = u(0, t) = 0,$$
$$u(\pi - 0, t) = u(\pi, t) = 0.$$

The terms of the series obtained by differentiating (1) with respect to t satisfy the inequality

$$|-kb_n n^2 e^{-n^2kt} \sin nx| < kMn^2 e^{-n^2kt_0} \quad \text{when } t \geqq t_0.$$

Since the series whose terms are $n^2e^{-n^2kt_0}$ also converges, according to the ratio test, that differentiated series is uniformly convergent for all $t \geqq t_0$. Hence series (1) can be differentiated termwise; that is,

$$\partial u \qquad \sum_1^\infty \frac{\partial}{\partial t}(b_n e^{-n^2kt} \sin nx)$$

In just the same way it follows that the series can be differentiated twice with respect to x whenever $t > 0$, and since each term of series (1) satisfies the heat equation, the function $u(x, t)$ must do so whenever $t > 0$ (Theorem 2, Chap. I).

It only remains to show that $u(x, t)$ satisfies the initial condition

(2) $$u(x, +0) = f(x) \qquad (0 < x < \pi).$$

This can be shown with the aid of a test, essentially due to Abel, for the uniform convergence of a series. At this time let us show how the test applies to the present problem, and defer the general statement of the test and its proof to the following chapter (see Theorem 1, Chap. VII).

For each fixed x $(0 < x < \pi)$, the series $\sum_1^\infty b_n \sin nx$ converges to $f(x)$. According to Abel's test, the new series formed by multiplying the terms of a convergent series by the corresponding members of a bounded sequence of functions of t, such as e^{-n^2kt}, whose functions never increase in value with n, converges uniformly with respect to t. Series (1) therefore converges uniformly with respect to t when $0 \leqq t \leqq t_1$, $0 < x < \pi$, for every positive t_1.

The terms of series (1) are continuous functions of t; hence the function $u(x, t)$ represented by that series is continuous with respect to t when $t \geqq 0$ and $0 < x < \pi$. Therefore

$$u(x, +0) = u(x, 0),$$

and condition (2) is satisfied because $u(x, 0) = f(x)$ $(0 < x < \pi)$.

The function $u(x, t)$ is now completely established as a solution of the boundary value problem (1)-(3), Sec. 45.

It is necessary to add to the statement of the problem some further restrictions as to the properties of continuity of the function sought, before we can prove that we have the only solution possible. We illustrate this by stating one complete form of the problem. For the sake of simplicity, we shall impose rather severe conditions of regularity on the functions involved.

A Complete Statement of the Problem. Let the function $u(x, t)$ be required to satisfy the heat equation and boundary conditions as given by equations (1) to (3), Sec. 45, in which the function $f(x)$ is now supposed continuous in the interval $0 \leqq x \leqq \pi$. We also assume that $f(0) = f(\pi) = 0$, and that $f'(x)$ is sectionally continuous in the interval $(0, \pi)$. In addition let it be required that $u(x, t)$ be continuous with respect to the two variables x, t together when $0 \leqq x \leqq \pi$, $t \geqq 0$, and that the derivative $\partial u/\partial t$ be continuous in the same manner whenever $t > 0$.

We can show that there is just one possible solution of this problem, and that solution is the function represented by series (1).

It was shown above that that function satisfies the heat equation and boundary conditions; also, that the series for $\partial u/\partial t$ converges uniformly with respect to x and t together when $0 \leqq x \leqq \pi$, $t \geqq t_0$ $(t_0 > 0)$. Since the terms of the derived series are continuous functions of x and t together, it follows that $\partial u/\partial t$ is continuous with respect to both variables together whenever $t > 0$, $0 \leqq x \leqq \pi$.*

The continuity of the function when $0 \leqq x \leqq \pi$ and $t \geqq 0$ follows again from our form of Abel's test. For the conditions on f ensure the uniform convergence of the series $\sum_1^\infty b_n \sin nx$. In this case the introduction of the factors e^{-n^2kt} into the terms of that series produces a series which is uniformly convergent with respect to x and t together, when $0 \leqq x \leqq \pi$, $0 \leqq t \leqq t_1$, for every positive t_1. Hence series (1) has this uniform convergence, and the continuity follows as before.

The function defined by series (1) therefore satisfies all the conditions of the problem. Of course, the derivative $\partial^2 u/\partial x^2$ is continuous in the same sense as $\partial u/\partial t$, since these two derivatives differ only by the factor k.

* Concerning the continuity of a series with respect to more than one variable, see the remarks preceding Theorem 1, Chap. VII.

It is not difficult to show that two distinct functions, satisfying all the requirements made upon $u(x, t)$ in the above statement of the problem, cannot exist. The complete statement and proof of this uniqueness theorem will be given later (Theorem 2, Chap. VII). If we accept this statement for the present, the only possible solution of our problem has been found.

47. Variations of the Problem of Temperatures in a Slab. With only slight modifications in the method, the temperature distribution can be found for the slab of Sec. 45 when the faces are subject to certain other conditions, or when the heat equation is modified.

a. One Face at Temperature A. To find the temperature $u(x, t)$ in a slab with initial temperature $f(x)$ when the face $x = 0$ is held at zero and the face $x = \pi$ at constant temperature A, a simple transformation can be used to obtain the result from that of Sec. 45.

Here $u(x, t)$ must be a solution of the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi,\ t > 0),$$
$$u(+0, t) = 0, \qquad u(\pi - 0, t) = A,$$
$$u(x, +0) = f(x).$$

It follows that the function

$$v(x, t) = u(x, t) - \frac{A}{\pi}x, \tag{1}$$

must satisfy the conditions

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \qquad (0 < x < \pi,\ t > 0),$$
$$v(+0, t) = 0, \qquad v(\pi - 0, t) = 0,$$
$$v(x, +0) = f(x) - \frac{A}{\pi}x.$$

This is the boundary value problem of Sec. 45 with $f(x)$ replaced by $f(x) - Ax/\pi$, so that its solution is

$$v(x, t) = \frac{2}{\pi}\sum_{1}^{\infty} e^{-n^2kt}\sin nx \int_0^\pi \left[f(x') - \frac{Ax'}{\pi}\right]\sin nx'\,dx'.$$

Substituting this for $v(x, t)$ in equation (1) and carrying out part of the integration, we obtain the following solution of the

problem:

$$u(x, t) = \frac{}{\pi} x$$

$$-\frac{2}{\pi} \qquad \left[(-1)^n \frac{A}{n} + \int_0^\pi f(x') \qquad x'\, dx'\right].$$

b. Insulated Faces. Find the temperature $u(x, t)$ in a slab with initial temperature $f(x)$ if the faces $x = 0$ and $x = \pi$ are thermally insulated.

Since the flux of heat through those faces is proportional to the values of $\partial u/\partial x$ there, the boundary value problem can be written

(2) $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi,\ t > 0),$$

$$: 0, \qquad \frac{\partial u(\pi - 0, t)}{\partial x} = 0 \qquad (t > 0),$$

(4) $$(x, +0) = f(x) \qquad (0 < x < \pi).$$

Setting $u = X(x)T(t)$, it is found that the functions

$$a_n e^{-n^2kt} \cos nx \qquad (n = 0, 1, 2, \cdots)$$

satisfy the homogeneous conditions (2) and (3). The infinite series of those functions satisfies condition (4) as well, provided the coefficients a_n are those in the Fourier cosine series corresponding to $f(x)$. So if $f(x)$ satisfies the conditions of our Fourier theorem, the solution of the problem is

(5) $$u(x, t) = \frac{1}{}$$

$$+ \frac{2}{\pi} \sum_1^\infty e^{-n^2kt} \cos nx \int_0^\pi f(x') \cos nx'\, dx'.$$

c. One Face Insulated. If the face $x = 0$ is held at temperature zero and the face $x = \pi$ is insulated, the problem can be reduced to one in which both faces are held at zero.

Let the slab be extended to $x = 2\pi$ with the face $x = 2\pi$ held at temperature zero, and let the initial temperature of the new slab be symmetric with respect to the plane $x = \pi$. Then, when $\pi < x < 2\pi$, the initial temperature is $f(2\pi - x)$, where $f(x)$ is the initial temperature of the original slab. In the

physical problem the symmetry indicates clearly that no heat will flow through the plane $x = \pi$. When the solution is found, it can be verified that $\partial u/\partial x = 0$ when $x = \pi$.

According to Prob. 1, Sec. 45, the temperature in the extended slab is

$$+ \int_{\pi}^{2\pi} f(2\pi - x') \sin \frac{nx'}{2} a$$

By substituting a new variable of integration in the second integral, this can be reduced to

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-m^2_n kt} \sin m_n x \int_0^{\pi} f(x') \sin m_n x' \, dx',$$

where $m_n = (2n - 1)/2$. When $0 \leqq x \leqq \pi$, this is the solution of problem *c*.

d. The Radiating Wire. Suppose the diameter of a wire or bar is small enough so that the variation of temperature over every cross section can be neglected. If the lateral surface is exposed to surroundings at temperature zero and loses or gains heat according to Newton's law, the heat equation takes the form

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - hu, \tag{6}$$

where x is the distance along the wire and h is a positive constant.

Newton's law of surface heat transfer is an approximate law of radiation and convection according to which the flux of heat through the surface of a solid is proportional to the difference between the temperature of the surface and that of the surroundings. It is generally valid only for small temperature differences; but it has the advantage over the more exact laws of being a linear relation. That the heat equation does take the form (6) when such surface heat transfer is present can be seen from the derivation of the heat equation (Sec. 9).

When the ends $x = 0$, $x = \pi$, of the radiating wire are kept at temperature zero and the initial temperature is $f(x)$, the temperature function can be found by the method of Sec. 45.

The result is

$$u(x, t) = e^{-ht}u_1(x, t), \tag{7}$$

where $u_1(x, t)$ is the function u in equation (8), Sec. 45.

When the ends are insulated, the result is

$$u(x, t) = e^{-ht}u_2(x, t), \tag{8}$$

where $u_2(x, t)$ is the function u in equation (5) above.

PROBLEMS

1. Derive the solution of the problem in Sec. 47*b* above when the faces are $x = 0$ and $x = L$.

Ans. $u =$

$$+ \frac{2}{L} \sum_{1} e^{-\frac{\quad}{L^2}} \cos \frac{n\pi x}{L} \int_0^L f(x') \cos \frac{n\pi x'}{L}\, dx'.$$

2. Show that the result of Prob. 1 can be completely established as a solution of the boundary value problem by the method of Sec. 46.

3. Solve the problem in Sec. 47*c* above for a slab of width L with the face $x = L$ insulated. It will be instructive to carry out the solution directly by obtaining particular solutions $u = XT$, without using the method of extension, noting the orthogonal functions generated by the differential equation in X and its boundary conditions (compare Sec. 25).

Ans. $$u = \frac{2}{L} \sum_{n=1}^{\infty} e^{-m_n^2 kt} \sin m_n x \int_0^L f(x') \sin$$

where $m_n = (n - 1/2)\pi/L$.

4. Derive formula (7).

5. Derive formula (8).

6. Use the substitution $v = ue^{ht}$ to simplify equation (6) and, by writing the boundary value problem in terms of $v(x, t)$, obtain formulas (7) and (8) from known results.

7. For a wire in which heat is being generated at a constant rate, while the lateral surface is insulated, the heat equation takes the form

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + B,$$

where B is a positive constant. If the ends $x = 0$ and $x = \pi$ are kept at temperature zero and the initial temperature is $f(x)$, set up the boundary value problem for $u(x, t)$ and solve it. Note the result when

$f(x) = Bx(\pi - x)/(2k)$. *Suggestion:* Apply the method used in Sec. 44 to reduce the nonhomogeneous differential equation to a homogeneous one.

Ans. $u = \dfrac{Bx}{2k}(\pi -$

$$\sum_{1} {}^{-n^2kt} \sin nx \int_0^\pi \left[\frac{Bx'}{2k}(x' - \pi) + f(x')\right] \sin nx'\, dx'.$$

8. Solve Prob. 7 when the end $x = \pi$ is insulated, instead of being kept at temperature zero.

9. A wire radiates heat into surroundings at temperature zero. The ends $x = 0$ and $x = \pi$ are kept at temperatures zero and A, respectively, and the initial temperature is zero. Set up and solve the boundary value problem for the temperature $u(x, t)$. *Suggestion:* Substitute $v = u + \psi(x)$, then determine ψ so that $k\psi'' - h\psi = 0$ and $\psi(0) = 0$, $\psi(\pi) = -A$

$$\textit{Ans.}\quad u = A\,\frac{\sinh x\sqrt{h/k}}{\text{inh}\, \pi\sqrt{h/k}} + \frac{2Ak}{\pi} e^{-ht} \sum_1^{c} \frac{n(-1)^n}{h + kn^2} e^{-n^2kt} \sin$$

10. The face $x = 0$ of a slab is kept at temperature zero and heat is supplied or extracted at a constant rate at the face $x = \pi$, so that $\partial u/\partial x = A$ when $x = \pi$. If the initial temperature is zero, derive the formula

$$u = Ax + \frac{2A}{\pi} \sum_1^\infty \frac{(-1)^n}{(n - \frac{1}{2})^2} e^{-(n-\frac{1}{2})^2t} \sin (n - \tfrac{1}{2})x,$$

for the temperatures in the slab, where the unit of time has been so chosen that $k = 1$.

48. Temperatures in a Sphere. Let the initial temperature in a homogeneous solid sphere of radius c be a function $f(r)$ of the distance from the center, and let the surface $r = c$ be kept at temperature zero. The temperature is then a function $u(r, t)$, of r and t only, and the heat equation in spherical coordinates becomes

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial^2(ru)}{\partial r^2}.$$

The boundary conditions are

$$u(c - 0, t) = 0 \qquad (t > 0),$$
$$u(r, +0) = f(r) \qquad (0 < r < c).$$

If we set $v(r, t) = ru(r, t)$, the boundary value problem here can be written

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial r^2},$$
$$v(+0, t) = 0, \qquad v(c - 0, t) = 0,$$
$$v(r, +0) = rf(r),$$

where the condition $v(+0, t) = 0$ is included because $u(r, t)$ must be bounded at $r = 0$. Except for the presence of r instead of x and $rf(r)$ instead of $f(x)$, this problem is that of the temperatures in a slab of width c. Hence the temperature formula for the sphere can be written at once (Prob. 1, Sec. 45). It is

$$_0 \quad 'f(r') \sin \frac{n\pi r'}{c}\, dr'.$$

PROBLEMS

1. Find the temperatures in a sphere if the initial temperature is zero throughout and the surface $r = c$ is kept at constant temperature A.

$$r, t) = A \quad \cdot \frac{2Ac}{\pi r} \quad \sin \cdot \frac{n\pi r}{}$$

2. Prove that the sum of the temperature function found in Prob. 1 and the function given by formula (1) above represents the temperature in a sphere whose initial temperature is $f(r)$ and whose surface is kept at temperature A.

3. An iron sphere with radius 20 cm., initially at the temperature 100°C. throughout, is cooled by keeping its surface at 0°C. Find to the nearest degree the temperature at its center 10 min. after the cooling begins, taking $k = 0.15$ c.g.s. unit. *Ans.* 22°C.

4. Solve Prob. 3, assuming that the sphere is made of concrete with $k = 0.005$ c.g.s. unit. *Ans.* 100°C.

5. The surfaces $r = b$ and $r = c$ of a solid in the form of a hollow sphere are kept at temperature zero. The initial temperature of the solid is $f(r)$ $(b < r < c)$. Derive the following formula for the temperatures $u(r, t)$ in the solid:

$$\sin \frac{n\pi(r - b)}{c - b},$$

where

$$c - \qquad \sin \frac{n\pi(r - b)}{} \, dr.$$

6. Show that when the surface of a sphere is insulated, the solution of the temperature problem no longer involves the expansion of $rf(r)$ in a Fourier series, but an expansion in a series of the functions $\sin \alpha_n r$, where α_n are the roots of the mixed equation $\tan \alpha c = \alpha c$. Show why these functions form an orthogonal set in the interval $0 < r < c$ (Sec. 25).

49. Steady Temperatures in a Rectangular Plate. Let $u(x, y)$ be the steady temperature at points in a plate with insulated faces, the edges of the plate being the lines (or planes) $x = 0$, $x = a$, $y = 0$, and $y = b$. Let three of the edges be kept at temperature zero and the fourth at a fixed temperature distribution. Then $u(x, y)$ is the solution of the following problem:

$$\text{(1)} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < a,\ 0 < y < b),$$

$$\text{(2)} \qquad u(+0, y) = 0, \qquad u(a - 0, y) = 0, \quad (0 < y < b),$$

$$\text{(3)} \qquad u(x, b - 0) = 0, \qquad u(x, +0) = f(x), \quad (0 < x < a).$$

Since special case (1) of the heat equation is also a case of Laplace's equation, the function $u(x, y)$ is also the potential in the rectangular region when the potential on the edges is prescribed by conditions (2) and (3). The region also may be considered as an infinitely long rectangular prism, or the right section of any prism in which the potential or steady temperature depends only upon x and y.

Setting $u = X(x)Y(y)$, the functions

$$\sin \frac{n\pi x}{a} \sinh \left[\frac{n\pi}{a}(y - C)\right] \quad (n = 1, 2, \cdots)$$

are found as solutions of (1) which satisfy conditions (2), for every constant C. If $C = b$, they also satisfy the first of conditions (3), and the series

$$u = \sum_{1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \left[\frac{n\pi}{a}(y - \right.$$

satisfies the nonhomogeneous condition in (3) provided

$$(0 < x < a).$$

According to the Fourier sine series, this is true if the coefficients A_n are determined so that

$$-A_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

So the formal solution of the problem can be written

$$u(x, y) = \frac{2}{a} \sum_1^{\infty} \frac{\sinh\{(n\pi/a)(b - y)\}}{\sinh\,(n\pi b/a)} \sin \frac{n\pi x}{a} \int_0^a f(x') \sin \frac{n\pi x'}{a}\, dx'$$

Our result can be completely established as a solution of the problem (1)-(3) by the method used in Sec. 46. But in this case let us defer that part of the discussion, along with a complete statement of the problem which ensures just one solution, until a later time when the necessary tests have been derived (Sec. 59).

PROBLEMS

1. Find the solution of the above problem if $u(x, y)$ is zero on all edges except $x = a$, and $u(a, y) = g(y)$.

$$\textit{Ans.}\quad u(x, y) = \frac{2}{b} \quad \frac{\sinh\,(n\pi x/b)}{\sinh\,(n\pi a/b)} \sin \frac{n\pi y}{b} \int_0^b g(y') \sin \frac{n\pi y'}{b},$$

2. When the temperature distributions on all four edges are given, show how the formula for the steady temperatures in the plate can be written by combining results already found.

3. What is the steady temperature at the center of a square plate with insulated faces, (*a*) if three edges are kept at 0°C. and the fourth at 100°C.; (*b*) if two adjacent edges are kept at 0°C. and the others at 100°C.? *Suggestion:* Superpose the solutions of like problems here to obtain the obvious case in which all four edges are kept at 100°C.

Ans. (*a*) 25°C.; (*b*) 50°C.

4. A square plate has its faces and its edge $y = 0$ insulated. Its edges $x = 0$ and $x = \pi$ are kept at temperature zero, and its edge $y = \pi$ at temperature $f(x)$. Derive the formula for its steady temperature.

$$\textit{Ans.}\quad u(x, y) = \frac{2}{-} \sum_1^{\infty} \cosh ny \quad . \qquad \int$$

5. Derive the formula for the electric potential $V(x, y)$ in the space $0 \leqq x \leqq L$, $y \geqq 0$, if the planes $x = 0$ and $x = L$ are kept at zero potential and the points of the plane $y = 0$ at the potential $f(x)$, if $V(x, y)$ is to be bounded as y becomes infinite.

$$\textit{Ans.}\quad V(x, y) = \frac{2}{L} \sum_1^{\infty} e^{-\frac{n\pi y}{L}} \sin \frac{n\pi x}{L} \int_0^L f(x') \sin$$

6. Find the electric potential in Prob. 5 if the planes $x = 0$ and $x = L$, instead of being kept at potential zero, are insulated, so that the electric force normal to those planes is zero; that is, $\partial V/\partial x = 0$. Also state this problem as a temperature problem.

Ans. $$V(x, y) = \frac{1}{L}\int_0^L f(x')\,dx'$$

$$\cos \frac{n\pi x}{\ } \int_0^{l} f(x') \cos \frac{n\pi x'}{l}\,dx'.$$

7. Solve Prob. 5 if the electric potential is zero on the plane $x \quad 0$ and the electric force normal to the plane $x = L$ is zero.

8. Find the steady temperatures in a semi-infinite strip whose faces are insulated and whose edges $x = 0$ and $x = \pi$ are kept at temperature zero, if the base $y = 0$ is kept at temperature 1 (Prob. 5).

Ans. $$u(x, y) = \qquad {}^{-y} \sin x \cdot \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5} \sin$$

9. In the power series expansion of $[\log(1 + z) - \log(1 - z)]$ $(|z| < 1)$, set $z = re^{i\varphi}$ and equate imaginary parts to find the sum of the series

$$S = r \sin \varphi + \tfrac{1}{3} r^3 \sin 3\varphi + \tfrac{1}{5} r^5 \sin 5\varphi + \cdots ;$$

also note that

$$\log [\rho(\cos \theta + i \sin \theta)] = \log (\rho e^{i\theta}) = \log \rho + i\theta,$$

and therefore show that

$$S = \frac{1}{2} \arctan \frac{2r \sin \varphi}{1 - r^2}.$$

Thus show that the answer to Prob. 8 can be written in closed form as follows:

$$u(x, y) = \frac{2}{\pi} \arctan \frac{\sin x}{\sinh y}.$$

Verify the answer in this form. Also trace some of the isotherms, $u(x, y) =$ a constant.

50. Displacements in a Membrane. Fourier Series in Two Variables. Let z represent the transverse displacement at each point (x, y) at time t in a membrane stretched across a rigid rectangular frame in the xy-plane. Let the boundaries of the rectangle be the lines $x = 0$, $x = x_0$, $y = 0$, and $y = y_0$. If the initial displacement z is a given function $f(x, y)$, and the membrane is released from rest after that displacement is made, the

boundary value problem in $z(x, y, t)$ is the following:

$$z(0, y, t) = 0, \qquad z(x_0, y, t) = 0;$$
$$z(x, 0, t) = 0, \qquad z(x, y_0, t) = 0;$$

In order that the product $z = X(x)Y(y)T(t)$ be a solution of equation (1), its factors must satisfy the equation

$$\frac{T''}{a^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

All three terms in this equation must be constant, since they are functions of x, y, and t separately. Write

$$\frac{X''}{X} = -\alpha^2, \qquad \frac{Y''}{Y} = -\beta^2;$$

then

$$T'' = -a^2(\alpha^2 + \beta^2)T.$$

The solutions of these three equations, for which $z = XYT$ satisfies all the homogeneous boundary conditions, are

$$X = \sin \alpha x, \qquad Y = \sin \beta y, \qquad T = \cos\left(a\sqrt{\alpha^2 + \beta^2}\,t\right),$$

where $\alpha = m\pi/x_0$, and $\beta = n\pi/y_0$ $(m, n = 1, 2, \cdots)$.

So the function

$$\sum_{m=1}\sum_{n=1}$$

satisfies equation (1) and all the boundary conditions, formally, provided the coefficients A_{mn} can be determined so that $z = f(x, y)$ when $t = 0$; that is, provided

(3) $$f(x, y) = \sum\sum A_{mn} \sin\frac{m\pi x}{x_0} \sin\frac{n\pi y}{y_0}$$
$$(0 \leqq x \leqq x_0,\ 0 \leqq y \leqq y_0).$$

By formally grouping the terms of the series, equation (3) can be written

$$f(x, y) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi y}{y_0}\right) \sin\frac{m\pi x}{\ }.$$

For each fixed y between zero and y_0 this series is the Fourier sine series of the function $f(x, y)$ of the variable x $(0 \leqq x \leqq x_0)$, provided the coefficients of $\sin (m\pi x/x_0)$ are those of the Fourier sine series. So equation (4) is true in general if

$$\sin \frac{n\pi y}{y_0} = \frac{2}{x_0} \int_0^{x_1} f(x', y) \sin \frac{m\pi x'}{x_0} dx'. \tag{5}$$

Again, using the formula for the Fourier sine coefficients of the functions $F_m(y)$, where

$$F_m(y) = \frac{2}{x_0} \int_0^{x_0} f(x', y) \sin \frac{m\pi x'}{x_0} dx' \qquad (0 \leqq y \leqq y_0),$$

expansion (5) is valid if

The series in equation (3) is then a Fourier sine series in two variables for $f(x, y)$ provided its coefficients have the values

$$\int_0^{y_0} dy \int_0^{x_0} f(x, y) \sin \frac{m\pi x}{x_0} \sin \frac{n\pi y}{} dx. \tag{6}$$

The formal solution of the membrane problem is then given by equation (2) with the coefficients defined by equation (6).

According to equation (2), the displacement z is not in general periodic in t, since the numbers $[(m^2/x_0^2) + (n^2/y_0^2)]^{\frac{1}{2}}$ do not change by multiples of any fixed number as the integers m and n change. Consequently the vibrating membrane, in contrast to the vibrating string, will not generally give a musical note. It can be made to do so, however, by giving it the proper initial displacement.

If, for instance, for any fixed integers M and N,

$$z(x, y, 0) = A \sin \frac{M\pi x}{x_0} \sin \frac{N\pi y}{y_0},$$

then the displacement (2) is given by a single term:

$$= A \cos \left(\pi a t \sqrt{\frac{M^2}{x_0^2} + \frac{N^2}{y_0^2}} \right) \sin \frac{M\pi x}{x_0} \sin \frac{N\pi y}{y_0}$$

In this case z is periodic in t with the period

$$(2/a)(M^2/x_0^2 -$$

PROBLEMS

1. Solve the above problem if the membrane starts from the position of equilibrium, $z = 0$, with an initial velocity at each point; that is,

$$\frac{\partial z(x, y, 0)}{\partial t} \quad F(x, y).$$

2. The four edges of a plate π units square are kept at temperature zero and the faces are insulated. If the initial temperature is $f(x, y)$, derive the following formula for the temperature $u(x, y, t)$:

$$u = \sum \qquad A_{mn} \exp\left[-k(m^2 + n^2)t\right] \sin mx \sin ny,$$

where

$$A_{mn} = \frac{4}{\pi^2} \int_0^\pi \sin ny\, dy \int_0^\pi f(x, y) \sin mx\, dx.$$

3. When $f(x, y) = Ax$, show that the solution of Prob. 2 is $u = u_1(x, t)u_2(y, t)$, where

n ,

1

$$u_2 = \frac{2}{\pi} \sum_1^\infty \frac{1 - (-1)^n}{n} e^{-n^2kt} \sin ny.$$

Show that u_1 and u_2 represent temperatures in cases of one-dimensional flow of heat with initial temperatures Ax and 1, respectively.

4. Solve Prob. 2 if, instead of being kept at temperature zero, the edges are insulated. Note the result when $f(x, y) = 1$.

5. If the faces $x = 0$, $x = \pi$, $y = 0$, $y = \pi$, $z = 0$, $z = \pi$ of a cube are kept at temperature zero and the initial temperature is given at each point $u(x, y, z, 0) = f(x, y, z)$, show that the temperature function is

$$u(x, y, z, t) = \sum_{m=1} \sum_{n=1} \qquad {}_{mnp} \sin mx \sin ny \sin pz\, e^{-k(m^2+n^2+p^2)t},$$

where

$$A_{mnp} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi f(x, y, z) \sin mx \sin ny \sin pz\, dx\, dy\, dz.$$

6. When $f(x, y, z) = 1$, show that the solution of Prob. 5 reduces to $u = u_2(x, t)u_2(y, t)u_2(z, t)$, where the function u_2 is defined in Prob. 3.

51. Temperatures in an Infinite Bar. Application of Fourier Integrals. Let the length of a homogeneous cylinder or prism be so great that it can be considered as extending the entire length of the x-axis. If the lateral surface is insulated and the initial temperature is given as a function $f(x)$ of position along the bar, the temperature $u(x, t)$ is the solution of the following boundary value problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0), \tag{1}$$

$$u(x, +0) = f(x) \qquad (-\infty < x < \infty). \tag{2}$$

Particular solutions of equation (1) which are bounded for all x and t ($t \geqq 0$) are found by the usual method to be

$$e^{-\alpha^2 kt}\cos[\alpha(x + C)], \tag{3}$$

where α and C are arbitrary constants. Any series of these functions, formed in the usual manner by taking α as multiples of a fixed number, would clearly reduce to a *periodic* function of x when $t = 0$. But $f(x)$ is not assumed periodic, and condition (2) is to be satisfied for all values of x; hence it is natural to try to use the Fourier integral here in place of the Fourier series.

Since function (3) is a solution of equation (1), so is the function

$$\frac{1}{\pi} f(x')e^{-\alpha^2 kt}\cos[\alpha(x' - x)],$$

where the parameters x' and α are independent of x and t. The integral of this with respect to these parameters,

$$u(x, t) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} f(x')e^{-\alpha^2 kt}\cos[\alpha(x' - x)]\,dx', \tag{4}$$

is then a solution of equation (1) provided this integral can be differentiated twice with respect to x and once with respect to t inside the integral signs.

When $t = 0$, the right-hand member of equation (4) becomes the Fourier integral corresponding to $f(x)$. Hence if $f(x)$ satisfies the conditions of the Fourier integral theorem, and if the function $u(x, t)$ defined by equation (4) is such that $u(x, 0) = u(x, +0)$, then

$$u(x, +0) = \tfrac{1}{2}[f(x + 0) + f(x - 0)],$$

and this is condition (2) at each point where $f(x)$ is continuous.

The solution of the problem is therefore given, at least formally, by equation (4). By inverting the order of integration and using the integration formula

$$\cos \qquad \exp\left[- \right.$$

equation (4) becomes

This can be still further reduced by using a new variable of integration ξ, where

$$\frac{x' - x}{\sqrt{4kt}},$$

this gives

(6) $$u(x, t) = \frac{1}{\sqrt{\pi}} \qquad x + 2$$

When $f(x)$ is bounded for all values of x and integrable in every finite interval, it can be shown that the function defined by equation (5) satisfies equation (1) and condition (2).* Under those conditions, then, the required solution is given either by equation (5) or by equation (6).

PROBLEMS

1. Derive the temperature function for the above bar if $f(x)$ is periodic with period 2π.

Ans. $$u(x, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx'$$

$$\frac{1}{\pi}\sum_{1}^{\infty} e^{-n^2kt}\int_{-\pi}^{\pi} f(x') \cos [n(x' -$$

2. If $f(x) = 0$ when $x < 0$, and $f(x) = 1$ when $x > 0$, show that the temperature formula for the infinite bar becomes, for $t > 0$,

$$1$$

$$2\sqrt{kt}$$

$$3(2\sqrt{kt})^3 \quad 5 \cdot 2!(2$$

* For a proof, see p. 31 of Ref. 1 at the end of this chapter.

3. One very thick layer of rock at 100°C. is placed upon another of the same material at 0°C. If $k = 0.01$ c.g.s. unit, find the temperatures to the nearest degree at points 60 cm. on each side of the plane of contact, 100 hr. after contact is made. *Ans.* 76°C.; 24°C.

52. Temperatures in a Semi-infinite Bar. If the bar of the foregoing section extends only along the positive half of the x-axis and the end $x = 0$ is kept at temperature zero, the boundary value problem in the temperature function $u(x, t)$ becomes the following:

$$\partial t$$

$$u(+0, t) = 0 \qquad (t > 0), \tag{2}$$

$$u(x, +0) = f(x) \qquad (x > 0). \tag{3}$$

The solution can be formed from the function $e^{-\alpha^2 kt} \sin \alpha x$, which satisfies conditions (1) and (2). Multiplying this by $(2/\pi)f(x') \sin \alpha x'$ and integrating with respect to the parameters α and x', which are independent of x and t, the function

$$u(x, t) = \frac{2}{\pi} \int_0^\infty e^{-\alpha^2 kt} \sin \alpha x \, d\alpha \int_0^\infty f(x') \sin \alpha x' \, dx' \tag{4}$$

is found. When $t = 0$, the integral on the right reduces to the Fourier sine integral of $f(x)$, which represents $f(x)$ when $0 < x < \infty$.

If we write

$$2 \sin \alpha x \sin \alpha x' = \cos [\alpha(x' - x)] - \cos [\alpha(x' + x)],$$

the integration formula used in the foregoing section can be applied to reduce formula (4) to the form

when $t > 0$. This can be written

$$u(x, t) = \frac{1}{\sqrt{\pi}} \left[\int_{-x}^{\infty} e^{-\xi^2} f(x + 2 \right. \tag{6}$$

$$- \int_{\frac{x}{2\sqrt{kt}}}^{\infty}$$

These results can also be found directly from those of the last section by making $f(x)$ there an odd function. Under the conditions stated in the preceding section, function (5) then satisfies all the conditions of the problem.

PROBLEMS

1. When $f(x) = 1$, prove that the temperature in the semi-infinite bar, or in a semi-infinite solid $x \geqq 0$, with its boundary $x = 0$ at zero, is

$$u(x, t) \quad \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}}$$

$$\frac{x^3}{3(2\sqrt{kt})^3} +$$

2. When the end $x = 0$ is kept at temperature A and the initial temperature of the bar is zero, show that

$$u(x, t) = A\left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-\xi^2}\, d\xi\right).$$

3. Show that when a semi-infinite solid initially at a uniform temperature throughout is cooled or heated by keeping its plane boundary at a constant temperature, the times required for any two points to reach the same temperature are proportional to the squares of their distances from the boundary plane.

4. Show that the function

satisfies all conditions of the boundary value problem consisting of equations (1) to (3) when $f(x) = 0$. Hence this function can be multiplied by any constant and added to the solution obtained above, to obtain as many solutions of that problem as we please. But also show that u_1 is not bounded at $x = t = 0$; this can be seen by letting x vanish while $x^2 = t$.

53. Further Applications of the Series and Integrals. Many other boundary value problems, arising frequently as problems in engineering or geology, can be solved by the methods of this chapter. A few will be stated at this point. The derivation of the results given here can be left as problems for the student.

a. Electric Potential between Parallel Planes. The plane $y = 0$ is kept at electric potential $V = 0$, and the plane $y = b$ at the

potential $V = f(x)$. Assuming that the space between those planes is free of charges, the potential $V(x, y)$ in the space is to be determined.

It can be shown that

$$(1)\quad V = \frac{1}{\pi}\int_0^\infty \frac{\sinh \alpha y}{\sinh \alpha b}\, d\alpha \int_{-\infty}^{\infty} f(x') \cos [\alpha(x' - x)]\, dx'.$$

Problems of this type are idealizations of problems arising in the design of vacuum tubes. They are also problems in steady temperatures, or steady diffusion, in solids; hence their applications are quite broad. The following problem is another of the same type.

b. Potential in a Quadrant. A medium free of electric charges has the planes $x = 0$ and $y = 0$ as its boundaries. If those planes are kept at electric potential $V = 0$ and $V = f(x)$, respectively, and if the potential $V(x, y)$ is bounded for all x and y $(x \geqq 0,\ y \geqq 0)$, the formula for $V(x, y)$ is to be found.

The result can be written, when $y > 0$, as

$$(2)\quad V = \frac{y}{\pi}\int_0^\infty f(x')\left[\frac{1}{y^2 + (x' - x)^2} - \frac{1}{y^2 + (x' + x)^2}\right] dx'.$$

When $f(x) = 1$, this formula becomes

$$(3)\qquad V = \frac{2}{\pi}\arctan\frac{x}{y}.$$

In this case the equipotential surfaces are the planes $x = cy$, where the constant c has the value $\tan (\pi V/2)$.

c. Angular Displacements in a Shaft. Let $\theta(x, t)$ be the angular displacement or twist in a shaft of circular cross section with its axis along the x-axis. If the ends $x = 0$ and $x = L$ of the shaft are free, the displacements $\theta(x, t)$ due to an initial twist $\theta = f(x)$ must satisfy the boundary value problem

$$\frac{\partial^2\theta}{\partial t^2} = a^2$$

$$\frac{t)}{} =$$

$$\theta(x, 0) = f(x),$$

where a is a constant.

The solution of this problem can be written

$$\theta = \frac{1}{2}a_0 + \sum^{\infty} a_n \cos\frac{n\pi x}{L} \cos\frac{n\pi at}{L}, \tag{4}$$

where a_n $(n = 0, 1, 2, \cdots)$ are the coefficients in the Fourier cosine series for $f(x)$ in the interval $(0, L)$.

d. The Simply Supported Beam. The differential equation for the transverse displacements $y(x, t)$ in a homogeneous beam or bar was given in Sec. 12. At an end which is simply supported or hinged, so that both the displacement and the bending moment are zero there, it can be shown that $\partial^2 y/\partial x^2$ must vanish as well as y. The displacements are to be found in a beam of length L with both ends simply supported, when the initial displacement is $y = f(x)$, and the initial velocity is zero.

The result is

$$y = \frac{2}{L}\sum^{\infty} \sin\frac{n\pi x}{L} \cos\frac{n^2\pi^2 ct}{L^2} \int_0^L f(x') \sin\frac{n\pi x'}{L}\, dx', \tag{5}$$

where c is the constant appearing in the differential equation.

PROBLEMS

1. Write the boundary value problem in Sec. 53*a* above, and derive solution (1).

2. Write the boundary value problem in Sec. 53*b*, and derive solution (2).

3. Obtain solution (3) from (2), and show that the function (3) satisfies all the conditions of the boundary value problem when $f(x) = 1$.

4. Derive the solution (4) of Sec. 53*c*. Also show how this formula can be written in closed form in terms of the even periodic extension of the function $f(x)$.

5. Set up the boundary value problem in Sec. 53*d*, and derive solution (5).

6. Derive the formula for the temperatures $u(x, t)$ in the semi-infinite solid $x \geqq 0$, if the initial temperature is $f(x)$ and the boundary $x = 0$ is kept insulated.

7. Find the formula for the displacements $y(x, t)$ in a string stretched between the points $(0, 0)$ and $(\pi, 0)$, if the string starts from rest in the position $y = f(x)$ and is subject to air resistance proportional to the velocity at each point. Let the unit of time be selected so that the equation of motion becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - 2h\frac{\partial y}{\partial t},$$

where h is a positive constant.

$$Ans. \quad y = e^{-ht} \sum_{1}^{\infty} b_n \left(\cos K_n t + \frac{h}{K} \sin K \right.$$

$$K_n = \sqrt{n^2 - h^2},$$

and b_n are the coefficients in the Fourier sine series for $f(x)$ in the interval $(0, \pi)$.

8. Let $V(r, \varphi)$ be the electric potential in the space inside the cylindrical surface $r = 1$, when the potential on this surface is a given function $f(\varphi)$ of φ alone. Note that $V(r, \varphi)$ must be periodic in φ with period 2π; it must also be a continuous function within the cylinder, since the space is supposed free of charges. Derive the following formula for $V(r, \varphi)$:

$$V = \tfrac{1}{2}a_0 + \sum_{1} r^n(a_n \cos n\varphi + b_n \sin n\varphi),$$

where a_n and b_n are the Fourier coefficients of $f(\varphi)$ for the interval $(-\pi, \pi)$.

9. In Prob. 8, suppose $f(\varphi) = -1$ when $-\pi < \varphi < 0$, and $f(\varphi) = 1$ when $0 < \varphi < \pi$, and show in this case that the potential formula can be written in the closed form

$$V = \frac{2}{\pi} \arctan \frac{2r \sin \varphi}{1 - r^2}$$

with the aid of the result found in Prob. 9, Sec. 49.

10. From the infinite solid cylinder bounded by the surface $r = c$ a wedge is cut by the axial planes $\varphi = 0$ and $\varphi = \varphi_0$. Find the steady temperatures $u(r, \varphi)$ in this wedge if $u = 0$ on the surfaces $\varphi = 0$ and $\varphi = \varphi_0$, and $u = f(\varphi)$ $(0 < \varphi < \varphi_0)$ on the convex surface of the wedge.

Ans. $u = \sum_{1}^{\infty} b_n (r/c)^{\frac{n\pi}{\varphi_0}} \sin (n\pi\varphi/\varphi_0)$, where b_n are the coefficients in the Fourier sine series for $f(\varphi)$ in the interval $(0, \varphi_0)$.

11. If in Prob. 10, $f(\varphi) = A$, where A is a constant, show that the formula for $u(r, \varphi)$ can be written in closed form with the aid of the result found in Prob. 9, Sec. 49.

REFERENCES

1. Carslaw, H. S.: "Mathematical Theory of the Conduction of Heat in Solids," 1921.
2. Ingersoll, L. R., and O. J. Zobel: "Mathematical Theory of Heat Conduction with Engineering and Geological Applications," 1913.
3. Timoshenko, S.: "Vibration Problems in Engineering," 1937.

CHAPTER VII

UNIQUENESS OF SOLUTIONS

54. Introduction. For the most part, our solutions of the boundary value problems in the last chapter were formal, in that we did not usually attempt to establish our result completely, or to find conditions under which the formula obtained represents the only possible solution. We shall develop a few theorems here which will furnish the reader interested in such matters with a mathematically complete treatment of many of our problems.

A multiplicity of solutions may actually arise when the problem is incompletely stated. Also, it is generally not a simple matter to transcribe a physical problem completely into its mathematical form as a boundary value problem. Consequently, the precise treatment of such problems is of practical as well as theoretical interest.

Our first theorem (Abel's test) enables us to establish the continuity of many of our results obtained in the form of series. The continuity property is useful both in demonstrating that our result is actually a solution of the boundary value problem, and in showing that it is the only solution.

The remaining theorems give conditions under which not more than one solution is possible. It will be evident that they can be applied only to specific types of problems. But no "general" uniqueness theorem exists in the theory of boundary value problems in partial differential equations, in the sense that the same theorem applies to temperature problems, potential problems, etc.

The uniqueness theorems given below are again special in that they require a high degree of regularity of the functions involved. But they will make possible a complete treatment of many of the problems considered in this book.

55. Abel's Test for Uniform Convergence of Series. We now establish a test for the uniform convergence of infinite series whose terms are products of specified types of functions. Applications of this test have already been made in the foregoing

chapter, to establish the continuity of the solution of a boundary value problem (Sec. 46).

The function represented by a uniformly convergent series of continuous functions is continuous. This is true regardless of the number of independent variables, as will be evident upon our recalling the method of proof for a single variable.* It is to be understood that the terms of the series are continuous with respect to all the independent variables taken together, in some region. The uniform convergence of the series in this region then ensures the same type of continuity of the sum of the series.

A sequence of functions $T_n(t)$ $(n = 1, 2, \cdots)$ is said to be *uniformly bounded* for all values of t in an interval if a constant K, independent of n, exists for which

for every n and all values of t in the interval. The sequence is *monotone with respect to n* if either

$$T_{n+1}(t) \leqq T_n(t) \tag{2}$$

for every t in the interval and for every n, or else

$$T_{n+1}(t) \geqq T_n(t) \tag{3}$$

for every t and n.

The following somewhat generalized form of a test due to Abel shows that when the terms of a uniformly convergent series are multiplied by functions $T_n(t)$ of the type just described, the new series is uniformly convergent.

Theorem 1. *The series*

$$\sum_{1}$$

converges uniformly with respect to the two variables x and t together, in a closed region R of the xt-plane, provided that (a) the series $\sum_{1}^{\infty} X_n(x)$ converges uniformly with respect to x in R, and (b) for all t in R the functions $T_n(t)$ $(n = 1, 2, \cdots)$ are uniformly bounded and monotone with respect to n.

Let S_n denote the partial sum of our series,

$$S_n(x, t) = X_1(x)T_1(t) + X_2(x)T_2(t) + \cdots + X_n(x)T_n(t).$$

* See, for instance, Sokolnikoff, "Advanced Calculus," p. 256, 1939.

We are to prove that, given any positive number ϵ, an integer N independent of x and t can be found such that

$$|S_m(x, t) - S_n(x, t)| < \epsilon \qquad \text{if } n > N,$$

for all integers $m = n + 1, n + 2, \cdots$, and for all x, t in the region R.

If we write

$$s_n(x) = X_1(x) + X_2(x) + \cdots + X_n(x),$$

then for every pair of integers m, n $(m > n)$, we have

(4) $S_m - S_n$

$= X_{n+1}T_{n+1} + X_{n+2}T_{n+2} + \cdots + X_mT_m$

$= (s_{n+1} - s_n)T_{n+1} + (s_{n+}$

$= (s_{n+1} -$

$\cdot_1 - T_m) + (s_m - s_n)T_m.$

Suppose now that the functions T_n are nonincreasing, with respect to n, so that they satisfy relation (2). Also let K be an upper bound of their absolute values, so that condition (1) is true. Then the factors $(T_{n+1} - T_{n+2})$, $(T_{n+2} - T_{n+3})$, $\cdots$, in equation (4) are non-negative, and $|T_m| < K$. Since the series $\sum_1^\infty X_n(x)$ is uniformly convergent, an integer N can be found for which

$$|s_{n+p} - s_n| < \frac{\epsilon}{3K} \qquad \text{when } n > N,$$

for all integers p, where ϵ is any given positive number and N is independent of x. For this choice of N it follows from equation (4) that

$$|S_m - S_n| < \frac{\epsilon}{3K}$$

$$+ \cdots + |T_m|] = \frac{\epsilon}{3K}[T_{n+1} - T_m + |T_m|];$$

and therefore

$$|S_m - S_n| < \epsilon, \quad \text{when } n > N \quad (m > n).$$

The proof of the theorem is similar when it is supposed that the functions T_n are of the nondecreasing type (3), with respect to n.

When the variable x is kept fixed, or when the functions $X_n(x)$ are constants, the theorem shows that the series with terms X_nT_n is uniformly convergent with respect to t. The only requirement on the series in X_n in this case is that the series shall converge.

Extensions of the theorem to the case in which the functions X_n involve the variable t as well as x, or where X_n and T_n are functions of any number of variables, become evident when it is observed that our proof rests on the uniform convergence of the X_n-series and the bounded monotone character, with respect to n, of the functions T_n.

56. Uniqueness Theorems for Temperature Problems. Let R denote the region interior to a solid bounded by a closed surface S, and let $R + S$ denote the closed region consisting of the points within the solid and upon its surface. If $u(x, y, z, t)$ represents the temperature at any point in the solid at time t, a rather general problem in the distribution of temperatures in an arbitrary solid is represented by the following boundary value problem;

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \varphi(x, y, z, t) \qquad (t > 0), \tag{1}$$

at all points (x, y, z) in R;

$$u = f(x, y, z) \tag{2}$$

in R, when $t = 0$;

$$u = g(x, y, z, t) \qquad (t > 0), \tag{3}$$

when (x, y, z) is on S.

This is the problem of determining the temperatures u in a solid with prescribed initial temperatures $f(x, y, z)$ and surface temperatures $g(x, y, z, t)$. A continuous source of heat, whose strength is proportional to $\varphi(x, y, z, t)$, may be present in the solid.

Suppose there are two solutions

$$u = u_1(x, y, z, t), \qquad u = u_2(x, y, z, t),$$

of this problem, where both u_1 and u_2 are continuous functions of x, y, z, t, together, in the region $R + S$ when $t \geqq 0$, while $\partial u_1/\partial t$, $\partial u_2/\partial t$, and all the derivatives of u_1 and u_2 once or twice

with respect to either x, y, or z are continuous functions when (x, y, z) is in $R + S$ and $t > 0$.

Since u_1 and u_2 satisfy each of the linear conditions (1) to (3), it follows at once that their difference w,

$$w(x, y, z, t) = u_1(x, y, z, t) - u_2(x, y, z, t),$$

satisfies the following linear homogeneous problem:

$$\frac{\partial w}{\partial t} = k\nabla^2 w \qquad \text{in } R \qquad (t > 0); \tag{4}$$

$$w = 0 \qquad \text{when } t = 0, \quad \text{in } R; \tag{5}$$

$$w = 0 \qquad \text{on } S \qquad (t > 0). \tag{6}$$

Moreover, w and its derivatives appearing in equation (4) must have the continuity properties required above of u_1 and u_2 and their derivatives.

We shall show now that w must vanish at all points of R for all $t > 0$, so that the two solutions u_1 and u_2 are identical. It follows that not more than one solution of the problem (1)-(3) can exist if the solution is required to satisfy the continuity conditions stated above.

Since the function w is continuous in $R + S$, the integral

$$J(t) = \frac{1}{2}\iiint_R w^2\,dV,$$

where $dV = dx\,dy\,dz$, is a continuous function of t when $t \geqq 0$. According to condition (5),

$$J(0) = 0.$$

In view of the continuity of $\partial w/\partial t$ when $t > 0$, we can write

$$\begin{aligned} J'(t) &= \iiint_R w\frac{\partial w}{\partial t}\,dV \\ &= k\iiint_R w\nabla^2 w\,dV \qquad (t > 0). \end{aligned}$$

Since the second derivatives of w with respect to each of the coordinates are continuous functions in $R + S$ when $t > 0$, we can use Green's theorem to write

$$\begin{aligned} &\iint_S w\frac{\partial w}{\partial n}\,dS \\ &= \iiint_R w\nabla^2 w\,dV + \iiint_R \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] dV. \end{aligned} \tag{7}$$

Here n is the outward-drawn normal to the surface S. But according to condition (6), $w = 0$ on S, and so

Since the integrand here is never negative,

$$J'(t) \leqq 0 \qquad \text{when } t > 0.$$

The mean-value theorem applies to $J(t)$ to give

$$J(t) - J(0) = tJ'(t_1) \qquad (0 < t_1 < t),$$

and since $J(0) = 0$, it follows that

$$J(t) \leqq 0 \qquad \text{whenever } t > 0.$$

However, the definition of the integral J shows that

$$J(t) \geqq 0 \qquad (t \geqq 0).$$

Therefore

$$J(t) = 0 \qquad (t \geqq 0);$$

and so the integrand w^2 of the integral J cannot be positive in R. Consequently

$$w(x, y, z, t) = 0$$

throughout $R + S$, when $t \geqq 0$.

This completes the proof of the following uniqueness theorem:

Theorem 2. *Let $u(x, y, z, t)$ satisfy these conditions of regularity: (a) it is a continuous function of x, y, z, t, taken together, when (x, y, z) is in the region $R + S$ and $t \geqq 0$; (b) those derivatives of u which are present in the heat equation* (1) *exist in R and are continuous in the same manner when $t > 0$. Then if u is a solution of the boundary value problem* (1)-(3), *it is the only possible solution satisfying the conditions* (a) *and* (b).

Our proof required only that the integral

$$\iint_S w \frac{\partial w}{\partial n}\, dS$$

in Green's theorem be zero or negative. The integral vanished, since $w = 0$ on S because of condition (3); but it is never positive if (3) is replaced by the condition

$$\frac{\partial u}{\partial n} + hu = g(x, y, z, t) \tag{8}$$

where h is a non-negative constant or function. So our theorem can be modified as follows:

Theorem 3. *The statement in Theorem 2 is true if boundary condition* (3) *is replaced by condition* (8), *or if* (3) *is satisfied on part of the surface* S, *and* (8) *on the remainder of* S.

The condition that u be continuous when $t = 0$ makes our uniqueness test somewhat limited. This condition is clearly not satisfied, for instance, if the initial temperature function is discontinuous in $R + S$, where the initial temperature on S is taken as the surface temperature.

If the regularity conditions (a) and (b) in Theorem 2 are added to the requirement that u must satisfy the heat equation and boundary conditions, our temperature problem will be *completely stated* provided it has a solution. For that will be the only possible solution.

57. Example. In the problem of temperatures in a slab with insulated faces and initial temperature $f(x)$ (Sec. 47b), suppose $f(x)$ is continuous when $0 \leqq x \leqq \pi$, and $f'(x)$ is sectionally continuous in that interval. Then the Fourier cosine series for $f(x)$ converges uniformly in the interval.

Let $u(x, t)$ denote the function defined by the series

$$\frac{1}{2}a_0 + \sum_1^\infty a_n e^{-n^2kt} \cos nx, \tag{1}$$

which was obtained in Sec. 47 as the formal solution, a_n being the coefficients in the Fourier cosine series for $f(x)$.

Series (1) converges uniformly with respect to x and t together when $0 \leqq x \leqq \pi$ and $t \geqq 0$, according to Theorem 1. In any interval throughout which $t > 0$, the series obtained by differentiating series (1) term by term, any number of times with respect to either variable, is uniformly convergent according to the Weierstrass M-test. It readily follows that $u(x, t)$ not only satisfies all the conditions of the boundary value problem (compare Sec. 46), but that it is also continuous when $0 \leqq x \leqq \pi$, $t \geqq 0$, and its derivatives $\partial u/\partial t$, $\partial^2 u/\partial x^2$ are continuous when $0 \leqq x \leqq \pi$, $t > 0$. That is, $u(x, t)$ satisfies our conditions of regularity.

The temperature problem for a slab is just the same as the problem for a cylindrical bar with its lateral surface insulated ($\partial u/\partial n = 0$); hence the region R can be considered here as a finite cylinder. Theorem 3 therefore applies, showing that the

function defined by series (1) is the only possible solution which satisfies the above regularity conditions.

PROBLEMS

1. In Prob. 7, Sec. 47, let $f(x)$ be continuous, and $f'(x)$ sectionally continuous, in the interval $(0, \pi)$, and suppose $f(0) = f(\pi) = 0$. Show that the solution found is the only one possessing the regularity properties stated above.

2. Make a complete statement of Prob. 8, Sec. 47, so that it has one and only one solution.

3. Establish the solution of Prob. 10, Sec. 47, and show that it is the only possible solution satisfying the regularity properties stated above.

58. Uniqueness of the Potential Function. A function of x, y, z is said to be *harmonic* in a closed region $R + S$, where S is a closed surface bounding a region R, if it is continuous in $R + S$ and if its second ordered derivatives with respect to x, y, and z are continuous in R and satisfy Laplace's equation there.

Let $U(x, y, z)$ be a harmonic function whose derivatives of the first order are continuous in $R + S$. Then since

$$\nabla^2 U = 0 \tag{1}$$

throughout R, Green's formula (7), Sec. 56, can be written as follows:

$$\iiint_R \Big[\Big($$

This formula is valid for our function U, even though we have not required the continuity of the second ordered derivatives of U in the closed region $R + S$. We shall not stop here to prove that, since $\nabla^2 U = 0$, this modification of the usual conditions in Green's theorem is possible.*

If $U = 0$ at all points on S, the first integral in equation (2) is zero, so the second integral must vanish. But the integrand of the second integral is clearly non-negative. It is also continuous in R. So it must vanish at all points of R; that is,

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0, \tag{3}$$

* The proof is not difficult. See, for instance, p. 119 of Ref. 1 at the end of this chapter.

so that U is constant in R. But U is zero on S and continuous in $R + S$, and therefore $U = 0$ throughout $R + S$.

Suppose that $\partial U/\partial n$, instead of U, vanishes on S; or, to make the condition more general, suppose

(4) on S,

where $h \geqq 0$, and h can be either a constant or function of x, y, and z. Then on S,

$$\frac{\partial U}{\partial n} \qquad \leqq 0,$$

so that the integral on the left in equation (2) is not positive. But the integral on the right is not negative. Both integrals therefore vanish and again condition (3) follows, so that U is constant throughout R.

Of course U may vanish over part of S and satisfy condition (4) over the rest of the surface, and our argument still shows that U is constant in R. In this case the constant must be zero.

Now suppose that the function $V(x, y, z)$, together with its derivatives of the first order, is continuous in $R + S$, and let its derivatives of the second order be continuous in R. Also let $V(x, y, z)$ be required to satisfy these conditions:

$$\nabla^2 V = f \tag{5}$$

when (x, y, z) is in R;

$$p\frac{\partial V}{\partial n} + hV = g \tag{6}$$

when (x, y, z) is on the surface S. The prescribed quantities f, p, h, and g may be functions of (x, y, z); but it is assumed that $p \geqq 0$ and $h \geqq 0$.

We have made boundary condition (6) general enough to include various cases of importance. When $p = 0$ on S, or on part of S, the value of V is assigned there; and when $h = 0$, the value of $\partial V/\partial n$ is assigned. Of course p and h must not vanish simultaneously.

If $V = V_1$ and $V = V_2$ are two solutions of this problem, then their difference,

$$U = V_1 - V_2,$$

satisfies Laplace's equation (1) in R, and the condition

on S. Since U is harmonic and has continuous derivatives of the first order in $R + S$, we have shown that U must be constant throughout $R + S$. Moreover, if $p = 0$ at any point of S, so that U vanishes there, then $U = 0$ throughout $R + S$.

We therefore have the following uniqueness theorem for problems in potential or steady temperatures, and other problems in which the differential equation is that of Laplace or Poisson.

Theorem 4. *Let the function $V(x, y, z)$ be required to satisfy these conditions of regularity: (a) it is continuous, together with its partial derivatives of the first order, in the region $R + S$; and (b) its derivatives $\partial^2 V/\partial x^2$, $\partial^2 V/\partial y^2$, and $\partial^2 V/\partial z^2$ are continuous functions in R. Then if V is a solution of the boundary value problem (5)-(6), it is the only possible solution satisfying the conditions of regularity, except for an arbitrary additive constant. If, in condition (6), $p = 0$ at any point of S, then the additive constant is zero and the solution is unique.*

It is possible to show that this theorem also applies when R is the infinite region outside the closed surface S, provided V satisfies the additional requirement that the absolute values of

$$\rho V, \qquad \rho^2 \frac{\partial V}{\partial x}, \qquad \rho^2 \frac{\partial V}{\partial y}, \qquad \rho^2 \frac{\partial V}{\partial z}$$

shall be bounded for all ρ greater than some fixed number, where ρ is the distance from the point (x, y, z) to any fixed point.* Since V is required to approach zero as ρ becomes infinite, the additive constant in this case is always zero. But note that even here S is a *closed* surface, so that this extension of our uniqueness theorem does not apply, for instance, to the infinite region between two planes or the infinite region inside a cylinder.

The regularity requirement (*a*) in Theorem 4 is quite severe. It will not be satisfied, for instance, in problems in which V is prescribed on the boundary as a discontinuous function, or as a function with a discontinuous derivative of the first order.

* For the proof, see Ref. 1.

For problems in which V is prescribed on the entire boundary S [that is, $p = 0$ in condition (6)] of the finite region R, it is possible to relax the conditions of regularity so as to require only the continuity of V itself in $R + S$. The derivatives of the first and second order are only required to be continuous in R. This follows directly from a remarkable theorem in potential theory: that if a function is harmonic in $R + S$, and not constant, its maximum and minimum values will be assumed at points on S, never in R.* But this uniqueness theorem is limited in its applications to boundary value problems, because it does not permit such a condition as $\partial V/\partial n = 0$ on any part of S, a condition which is often present or implied in the problem. This will be illustrated in the example to follow.

59. An Application. To illustrate the use of the theorem in the preceding section, consider the problem, in Sec. 49, of determining the steady temperature $u(x, y)$ in a rectangular plate with three edges kept at temperature zero and with an assigned temperature distribution on the fourth. The faces of the plate are kept insulated. For the purpose of illustration it will be sufficient to consider here only the case of the square plate with edge π units long. We also observe that as long as $\partial u/\partial n = 0$ on the faces, the thickness of the plate does not affect the problem.

We may as well consider this as a problem in the potential $V(x, y)$ in the finite region R bounded by the planes $x = 0$, $x = \pi$, $y = 0$, $y = \pi$, and any two planes $z = z_1$, $z = z_2$. Then our boundary value problem can be written

$$\frac{\partial^2 V}{\partial x^2} + \frac{\quad}{\partial y^2} \tag{1}$$

$$V(0, y) = 0, \qquad V(\pi, y) = 0, \tag{2}$$

$$V(x, 0) = f(x), \qquad V(x, \pi) = 0, \tag{3}$$

and of course, $\partial V/\partial z = 0$ on $z = z_1$ and $z = z_2$.

The given function $f(x)$ will be required here to be continuous, together with its first derivative, in the interval $(0, \pi)$. It is also supposed that $f''(x)$ is sectionally continuous in that interval; and finally, we require $f(x)$ to satisfy the conditions

$$f(0) = f(\pi) = 0.$$

* The proofs of these theorems will be found quite interesting, and not difficult to follow. See Refs. 1 and 2.

Then, according to our theory of Fourier series, the sine series for $f(x)$,

$$\sum_{1}^{\infty} b_n \sin nx \qquad \left[b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx\right] \tag{4}$$

converges uniformly, and so does the cosine series for $f'(x)$,

$$\sum^{\infty} nb_n \cos nx, \tag{5}$$

obtained by differentiating the sine series termwise. In demonstrating the uniform convergence of the Fourier series in Sec. 38, however, we proved that the series of the constants $\sqrt{a_n^2 + b_n^2}$ converges. In the case of series (5), in which the sine coefficients are zero and the cosine coefficients are nb_n, this means that the series

$$\sum |nb_n|$$

is convergent. Since the absolute values of the terms of the series

$$\sum_{1}^{\infty} nb_n \sin nx \tag{6}$$

are not greater than $|nb_n|$, it follows from the Weierstrass test that the series (6) also converges uniformly.

In addition to the conditions (1) to (3), let the unknown function $V(x, y)$ be required to satisfy the regularity conditions (a) and (b) of Theorem 4. That is, V, $\partial V/\partial x$, and $\partial V/\partial y$ must be continuous in the closed region $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$, while $\partial^2 V/\partial x^2$ and $\partial^2 V/\partial y^2$ are required to be continuous at all interior points of the region. We shall call this *a complete statement of the problem* of determining the function $V(x, y)$. For according to Theorem 4, this problem cannot have more than one solution, and we shall now prove that it does have a solution.

The series derived in Sec. 49 as the formal solution of our problem can be written here as

$$\sum_{1}^{\infty} b_n \frac{\sinh n(\pi - y)}{\sinh n\pi} \sin nx. \tag{7}$$

Let us show that this represents a function $\psi(x, y)$ which satisfies all the requirements made upon V in the complete statement of our problem, so that $V = \psi\,(x, y)$ is the unique solution of that problem.

To examine the uniform convergence of series (7), let us first show that the sequence of the functions

$$\frac{\qquad}{\sinh n\pi}, \tag{8}$$

which appear as factors in the terms of that series, is monotone nonincreasing as n increases, for every y in the interval $0 \leqq y \leqq \pi$. This is evident when $y = 0$ and $y = \pi$. It is true when $0 < y < \pi$, provided that the function

$$T(t) = \frac{\sinh bt}{\sinh at}$$

always decreases in value as t grows, when $t > 0$ and $a > b > 0$.

Now

$$\begin{aligned} 2T'(t) \sinh^2 at &= 2b \sinh at \cosh bt - 2a \sinh bt \cosh at \\ &= -(a - b) \sinh (a + b)t + (a + b) \sinh (a - b)t \\ &= -(a^2 - b^2)\left[\frac{\sinh (a + b)t}{a + b} - \frac{\sinh (a - b)t}{a - b}\right] \\ &\qquad \overline{\;\;}_{0} \;\; [(a + b)^{2n} - (a - b)^{2n}] \frac{t^{2n+1}}{(2n \;\; 1} \end{aligned}$$

The terms of this series are positive, so that

$$T'(t) < 0,$$

and $T(t)$ decreases as t increases. Therefore functions (8) never increase as n grows.

Likewise the functions

$$\frac{\cdot(\pi \qquad y)}{\sinh n\pi} \qquad (0 \leqq y \leqq \pi) \tag{9}$$

never increase in value when n grows; because the squares of these functions can be written as the sum

$$\frac{1}{\sinh^2 n\pi} + \frac{\sinh^2 n(\pi - y)}{\sinh^2 n\pi},$$

and as n grows the first term of the sum clearly decreases, while the second was just shown to be nonincreasing.

Functions (8) are clearly positive and not greater

their squares.

Therefore the sequence of functions (8), or of functions (9), can be used in our form of Abel's test for uniform convergence. So, from the uniform convergence of the sine and cosine series (4), (5), and (6), when $0 \leqq x \leqq \pi$, we conclude not only that our series (7) converges uniformly with respect to x, y in the region $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$, but also that this uniform convergence holds true for the series

$$nb_n \frac{\sinh n(\pi - y)}{\sinh n\pi} \cos nx,$$

1

obtained by differentiating series (7) with respect to x, and for the series

$$\frac{n(\pi -}{\sinh}$$

obtained by differentiating series (7) with respect to y.

Consequently series (7) converges to a function $\psi(x, y)$ which, together with its partial derivatives of the first order, is continuous in the closed region $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$. The function ψ clearly satisfies boundary conditions (2) and (3).

When differentiated twice with respect to either x or y, the terms of series (7) have absolute values not greater than the numbers

$$n^2|b_n| \frac{\sinh n(\pi - y_0)}{\sinh n\pi} \tag{11}$$

for all x and y in the region $0 \leqq x \leqq \pi$, $y_0 \leqq y \leqq \pi$, where y_0 is any positive number less than π. Since the series of the constants (11) converges, the series of the second derivatives of the terms of series (7) converges uniformly in the region specified. Hence series (7) can be differentiated termwise in this respect whenever $0 < y < \pi$; also the derivatives $\partial^2\psi/\partial x^2$, $\partial^2\psi/\partial y^2$ are continuous whenever $0 \leqq x \leqq \pi$, $0 < y \leqq \pi$.

Thus $\psi(x, y)$ satisfies the regularity conditions. It only remains to note that it satisfies Laplace's equation (1) in R. This is true because the terms of series (7) satisfy that equation, and series (7) is termwise differentiable twice with respect to x and to y in R, so that Theorem 2, Chap. I, applies.

The only solution of our completely stated problem is therefore

$$- \sum_{1} b_n \frac{\sinh n(\pi - y)}{\sinh n\pi} \sin nx.$$

In particular, note that we have shown that our complete problem, which includes the condition that $\partial V/\partial z = 0$ on the boundaries $z = z_1$ and $z = z_2$, has no solution which varies with z. In the formal treatment of the problem given earlier, the absence of the variable z was regarded as physically evident. In the present section we have omitted the term $\partial^2 V/\partial z^2$ in Laplace's equation, and at other times have neglected writing the variable z, only as a matter of convenience.

PROBLEMS

1. Show that the formal solution found in Sec. 49 can be completely established as one possible solution of the boundary value problem written there, provided the function $f(x)$ is sectionally continuous in the interval $(0, a)$ and has one-sided derivatives there, and $f(x)$ is defined to have the value $[f(x + 0) + f(x - 0)]/2$ at each point x of discontinuity $(0 < x < a)$.

2. Make a complete statement of the boundary value problem for the steady temperatures in a square plate with insulated faces, if the edges $x = 0$, $x = \pi$, and $y = 0$ are insulated, and the edge $y = \pi$ is kept at the temperature $u = f(x)$. Assume that $f''(x)$ is continuous when $0 \leqq x \leqq \pi$, and that $f'(0) = f'(\pi) = 0$. Show that your problem has the unique solution

$$u = \frac{1}{2} a_0 + \frac{2}{\pi} \sum_{1}^{\infty} a_n \frac{\cosh ny}{\cosh n\pi} \cos nx \qquad \left[a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \text{ cc} \right.$$

3. Establish the result found in Prob. 5, Sec. 49, as a solution (but not as the only possible one) of the boundary value problem, when the function $f(x)$ there is represented by its Fourier sine series.

4. In Prob. 8, Sec. 53, let the infinite cylinder be replaced by a finite cylinder bounded by the surfaces $r = 1$, $z = z_1$, $z = z_2$, on the last two of which $\partial V/\partial z = 0$. Also let the periodic function $f(\varphi)$ have a con-

tinuous second derivative. Then show that the result found there is actually a solution, and that it is the only possible solution of the problem satisfying our conditions of regularity.

REFERENCES

1. Kellogg, O. D.: "Foundations of Potential Theory," 1929.
2. Courant, R., and D. Hilbert: "Methoden der Mathematischen Physik," Vol. 2, 1937.

CHAPTER VIII

BESSEL FUNCTIONS AND APPLICATIONS

60. Derivation of the Functions $J_n(x)$. Any solution of the differential equation

known as *Bessel's equation*, is called a *Bessel function* or cylindrical function. It will be shown later on how this equation arises in the process of obtaining particular solutions of the partial differential equations of physics, written in cylindrical coordinates. We shall let the parameter n be any real number.

A particular solution of Bessel's equation in the form of a power series multiplied by x^p, where p is not necessarily an integer, can always be found. Let a_0 be the coefficient of the first nonvanishing term in such a series, so that $a_0 \neq 0$. Then our proposed solution has the form

$$y = x^p \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_j x^{p+j}. \tag{2}$$

If the series here can be differentiated termwise, twice, the coefficients a_j can be determined so that the series is a solution of equation (1). For upon differentiating and substituting in equation (1), we obtain the equation

$$\sum_{j=0}^{\infty} [(p+j)(p+j-1) + (p+j) + (x^2 - n^2)]a_j x^{p+j} = 0.$$

Dividing through by x^p and collecting the coefficients of the powers of x, we can write the equation in the form

$$(p^2 - n^2)a_0 + [(p+1)^2 - n^2]a_1 x + \sum_{j=2}^{\infty} \{[(p+j)^2 - n^2]a_j + a_{j-2}\}x^j = 0.$$

This is to be an identity in x, so that the coefficient of each power of x must vanish. The constant term vanishes only if $p = \pm n$. The second term vanishes if $a_1 = 0$; and the coefficients of the second and higher powers of x all vanish if

$$[(p + j)^2 - n^2]a_j + a_{j-2} = 0 \qquad (j = 2, 3, \cdots);$$

that is, if

$$(p - n + j)(p + n + j)a_j = -a_{j-2} \qquad (j = 2, 3, \cdots).$$

This is a *recursion formula* for a_j, giving each coefficient in terms of one appearing earlier in the series.

Let us make the choice

$$p = n,$$

so that the recursion formula becomes

$$j(2n + j)a_j = -a_{j-2} \qquad (j = 2, 3, \cdots). \tag{3}$$

Since $a_1 = 0$, it follows that $a_3 = 0$; hence $a_5 = 0$, etc.; that is,

$$a_{2k-1} = 0 \qquad (k = 1, 2, \cdots), \tag{4}$$

provided n is such that $2n + j \neq 0$ in formula (3). But even if $2n + j$ does vanish for some integer j, coefficients (4) still satisfy formula (3). Since this is all that is required to find a solution, we can take all the coefficients a_{2k-1} as zero regardless of the value of n.

Replacing j by $2j$ in formula (3), we can write

$$-1 \qquad a_{2j-2} \qquad (j = 1, 2, \cdots$$

provided *n is not a negative integer*. Replacing j by $j - 1$ here, we have

$$-1$$

so that

$$a_{2j} = \overline{2^4 j(j -}$$

Continuing in this manner, it follows that

$$a_{2j} = (-1)^k a_{2j-2k}/[2^{2k} j(j - 1) \cdots (j - k + 1)$$
$$(n + j)(n + j - 1) \cdots (n + j -$$

so that when $k = j$, we have the formula for a_{2j} in terms of a_0:

$$+ j)(n + j - 1) \cdots (n + 1) \qquad (j = 1, 2, \cdots).$$

The coefficient a_0 is left as an arbitrary constant. Let its value be assigned as follows:

$$a \qquad 1$$

Recalling that the Gamma function has the factorial property

$$= \Gamma(k + 1),$$

it follows that

$$(n + j)(n + j - 1) \cdots (n + 2)(n + 1)\Gamma(n + 1) = \Gamma($$

Our formula for a_{2j} can therefore be written

(5) $$a_{2j} = \frac{(-1)^j}{j!\Gamma(n + j + 1)2^{n+2j}} \qquad (j = 0, 1, 2, \cdots),$$

where $j! = 1$ if $j = 0$.

The function represented by series (2) with coefficients (4) and (5) is called a *Bessel function of the first kind of order n:*

(6)

$$\frac{}{2^n\Gamma(n + 1)} \Big[1 - \frac{}{2(2n + 2)} + \cdots$$

The series in brackets is absolutely convergent for all values of x, according to the ratio test. It is a power series, so that the termwise differentiation employed above is valid, and hence function (6) is a solution of Bessel's equation. Of course, when n is not a positive integer, $J_n(x)$ or its derivatives beyond a certain order will not exist at $x = 0$, because of the factor x^n.

61. The Functions of Integral Orders. When $n = 0$, the important case of the function of *order zero* is obtained:

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{\quad 4^2}$$

When n is a negative integer, the choice $p = -n$ can be made and the recursion formula (3) of the last section gives the coefficients just as before. If, in this case,

$$a_0 = \frac{2^n}{\Gamma(-n+1)},$$

the solution of Bessel's equation will be found to be

$$\sum_{j=0}$$

Now if we define $1/\Gamma(p)$ to be zero when $p = 0, -1, -2 \cdots$, formula (6) of the last section can be used to define a function $J_n(x)$ even when n is a negative integer. For if $n = -m$, where m is a positive integer, that formula becomes

Summing with respect to k, where $k = -m + j$, this can be written

$$J_{-m}(x) = (-1)^m \quad {}_{k=0}$$

But the last series represents $J_m(x)$; hence for functions of integral order,

$$(2) \qquad J_{-m}(x) = (-1)^m J_m(x) \qquad (m = 1, 2, 3, \cdots).$$

According to solution (1) now, the function $y = (-1)^n J_n(x)$, and hence the function $y = J_n(x)$ is a solution of Bessel's equation when n is a negative integer; *hence the function defined by equation* (6), *Sec.* 60, *is a solution for every real* n.

When n is neither a positive nor a negative integer, nor zero, it can be shown that the particular solution $J_{-n}(x)$ obtained by taking $p = -n$ is not a constant times the solution $J_n(x)$; hence the general solution of Bessel's equation in this case is

$$y = AJ_n(x) + BJ_{-n}(x),$$

where A and B are arbitrary constants.

When n is an integer, the general solution of Bessel's equation is

$$y = AJ_n(x) + BY_n(x),$$

where $Y_n(x)$ is a Bessel function of the second kind of order n. These functions will not be used here. For their derivation and properties, as well as for a more extensive treatment of the theory of Bessel functions of the first kind than we can give here, the reader should consult the references at the end of this chapter.

There are several other ways of defining the functions $J_n(x)$. When n is zero or a positive or negative integer, the generating function $\exp\left[\frac{1}{2}x(t - 1/t)\right]$, is often used for this purpose. By multiplying the two series

$$= e$$

it can be seen that

$$(3)\quad \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-}^{\infty}$$

$$= J_0(x)$$

for all values of x and t except $t = 0$. Hence $J_n(x)$ can be defined as the coefficients in this expansion. It is on the basis of this definition that the above choice of the constant a_0 was made.

PROBLEMS

1. Prove that

2. Prove that

$$\sqrt{}\ \sin x.$$

3. Derive solution (1) when n is a negative integer.

4. Carry out the derivation of formula (3).

5. Show that, for every n,

62. Differentiation and Recursion Formulas. By differentiating the series

(1) $$\infty \quad \frac{(-1)^j}{\quad} \left(\frac{x}{2}\right)$$

it follows that

(2) $$xJ'_n(x) = \quad 2j) \left(\frac{x}{2}\right.$$

$$= nJ_n(x) + \quad \frac{(-1)^j}{(j-1)!\Gamma(n+j+1)}\left(\frac{x}{2}\right)$$

$$= nJ_n(x) - \quad \frac{\quad}{k \quad 2)}\left(\frac{x}{2}\right)^{n+1+2k}$$

That is,

(3) $$- xJ_{n+1}(x).$$

Similarly, if we write $n + 2j = 2(n + j) - n$ in the second member of equation (2), and replace $\Gamma(n + j + 1)$ by $(n + j)\Gamma(n + j)$, we obtain the relation

$$-1+2j$$

that is,

(4) $$xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x).$$

Elimination of $J_n(x)$ between equations (3) and (4) gives the formula

(5) $$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x);$$

and the elimination of $J'_n(x)$ between the same two equations gives the formula

(6) $$2n \; J_n(x)$$

The *recursion formula* (6) gives the function $J_{n+1}(x)$ of any order in terms of the functions $J_n(x)$ and $J_{n-1}(x)$ of lower orders.

By multiplying equation (4) by x^{n-1}, and equation (3) by we can write these formulas, respectively, as

$$\frac{d}{\quad}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx}[x^{-n}J_n(x)] = -$$

The following consequences of the above formulas should be noted:

(7) $$\int_0^x rJ_0(r)\,dr = xJ_1(x).$$

PROBLEMS

1. Obtain formula (7) above.
2. Prove that

$$x^2J_n''(x) = [n(n-1) - x^2]J_n(x) + xJ_{n+1}(x$$

3. With the aid of Probs. 1 and 2, Sec. 61, prove that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{}} \Big(\sin x$$

4. When n is half an odd integer, show that $J_n(x)$ can always be written in closed form in terms of $\sin x$, $\cos x$, and powers of $1/\sqrt{x}$.

63. Integral Forms of $J_n(x)$. Let us first recall that the Beta function is defined by the formula

$$B(n + \tfrac{1}{2}, j + \tfrac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \sin^{2n}\theta \cos^{2j}\theta\,d\theta \quad (n > -\tfrac{1}{2}, j > -\tfrac{1}{2}).$$

Let j be zero or a positive integer. Then

$$B(n + \tfrac{1}{2}, j + \tfrac{1}{2}) = \int_0^{\pi} \sin^{2n}\theta \cos^{2j}\theta\,d\theta \quad (n > -\tfrac{1}{2}).$$

This function is given in terms of the Gamma function by the formula

$$\overline{\Gamma(n+j+1)},$$

and as a consequence we shall be able to write the general term of our series for $J_n(x)$ in terms of trigonometric integrals.

Our formula for $J_n(x)$ can be written

$$\left(\overline{2}\right) \sum_{j=0} \overline{2^{2j}j!\Gamma(n+j+1)}$$

Now

$$1 \qquad \_\ \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdots(j-\tfrac{1}{2})\Gamma(\tfrac{1}{2})$$

$$n$$

$$\overline{(2j!)\Gamma(n+j+1)\Gamma(\tfrac{1}{2})} - \overline{(2j)!\Gamma(\tfrac{1}{2})\Gamma(n}$$

Therefore

$$(-1)^j \qquad\qquad\qquad s^{2j}\,\theta\, d\theta,$$

$$_{j=0}$$

where

(2) $$C_n = \overline{\Gamma(\tfrac{1}{2})\Gamma(n}$$

When $n \geqq 0$, the series

$$\sum_{j=0}^{\infty} \frac{(-1)}{(2j)!} x^{2j} \sin^{2n}\theta \cos^{2j}\theta$$

converges uniformly with respect to θ in the interval $(0, \pi)$; because the absolute values of its terms are not greater than the corresponding terms of the convergent series

and the terms here are independent of θ. The first series can therefore be integrated termwise with respect to θ over the interval $(0, \pi)$. In other words, the integral sign in formula (1) can be written either before or after the summation sign. Therefore,

$$= C_n \int_0^{\pi} \quad {}^{2n}\,\theta \sum_{j=0}$$

Since the series in the integrand represents $\cos(x \cos\theta)$,

(3) $$J_n(x) = C_n \int_0^{\pi} \sin^{2n}\theta \cos(x\cos\theta)\, d\theta,$$

where C_n is defined by formula (2).

Formula (3) gives one of *Lommel's integral forms* of $J_n(x)$.

Although the above derivation holds only for $n \geqq 0$, form (3) is valid when $n > -\frac{1}{2}$. This can be shown by writing the first term in series (1) separately and integrating the remaining terms by parts to obtain the equation

$$J_n(x) = C_n \left[\int_0^{\pi} \sin^{2}\,\theta\, d\theta \right.$$

$$1^{2n+2}\,\theta \cos^{2j-2}\theta$$

if $n > -\frac{1}{2}$. Here again the second integral sign can be written before the summation sign, and the series in the integrand can be seen to represent the function

$$-\frac{\sin^{2n+1}\theta}{}\frac{d}{d\theta}\left(\frac{\cos(x\cos\theta) - 1}{\cos\theta}\right).$$

The details here are left to the reader. Integrating this by parts and adding the first integral in brackets gives formula (3) for $n > -\frac{1}{2}$.

When $n = 0$, formula (3) becomes

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta.$$

When $n = 0, 1, 2, \cdots$, the following integral form is valid:

$$(4)\quad J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta \qquad (n = 0, 1, 2, \cdots).$$

This is known as *Bessel's integral form*. By writing the integrand as

$$\cos n\theta\cos(x\sin\theta) + \sin n\theta\sin(x\sin\theta),$$

it can be seen that formula (4) reduces to

$$(5)\quad J_n(x) = \frac{1}{\pi}\int_0^\pi \cos n\theta\cos(x\sin\theta)\,d\theta \text{ if } n = 0, 2, 4, \cdots;$$

$$(6)\quad\qquad = \frac{1}{\pi}\int_0^\pi \sin n\theta\sin(x\sin\theta)\,d\theta \text{ if } n = 1, 3, 5,$$

These forms can be obtained from formula (3), Sec. 61. By substituting $t = e^{i\theta}$ in that formula, we find that

$$(7)\quad \cos(x\sin\theta) + i\sin(x\sin$$

$$= J_0(x) + 2\sum^{\infty} J_{2n}(x)\cos 2n\theta \cdot \qquad (x)\sin(2n -$$

Equating real parts and imaginary parts separately here, and multiplying the resulting equations by $\sin n\theta$ or $\cos n\theta$ and integrating, using the orthogonality of these functions in the interval $0 < \theta < \pi$, we get formulas (5) and (6). Formula (4) follows by the addition of the right-hand members of formulas (5) and (6). The details are left for the problems.

From formula (4) the important *property of boundedness*

$$|J_n(x)| \leqq 1 \qquad (n = 0, 1, 2, \cdots),$$

follows at once. It also follows from the same formula that *each derivative of $J_n(x)$ is bounded for all x:*

$$\leqq 1 \qquad (n = 0, 1, 2, \cdots; k = 1, 2, \cdots).$$

According to formula (5),

$$2J_{2n}(x) = a_{2n} \qquad (n = 1, 2, \cdots),$$

where a_n denotes the coefficients in the Fourier cosine series, with respect to θ, of the function $\cos (x \sin \theta)$. Similarly if b_n denotes the coefficients in the Fourier sine series of the function $\sin (x \sin \theta)$, formula (6) shows that

$$2J_{2n-1}(x) = b_{2n-1} \qquad (n = 1, 2, \cdots).$$

Since the Fourier coefficients of every bounded integrable function tend to zero as n tends to infinity, it follows that *for every x the Bessel functions of integral orders have the property*

$$\lim_{n \to \infty} J_n(x) = 0.$$

As to the behavior of the functions $J_n(x)$ for large values of x, it can be shown that

(8) $$\lim_{x \to \infty} J_n(x) = 0 \qquad (n = 0, 1, 2, \cdots).$$

The proof is left to the problems.

PROBLEMS

1. Use the Lommel integral form of $J_n(x)$ to prove that

$$= \sqrt{\frac{}{}} \sin x.$$

2. Prove in different ways that

$$J_n(-x) = (-1)^n J_n(x) \qquad (n = 0, 1, 2, \cdots),$$

and hence that $J_n(x)$ is an even or odd function of x according as n is an even or odd integer. Also deduce that

$$J_{2n-1}(0) = 0 \qquad (n = 1, 2, \cdots).$$

3. Prove in different ways that

$$J_0(0) = 1.$$

4. Obtain formula (7) above by the method indicated there, and follow the process outlined to derive Bessel's integral forms (5) and (6), and thence (4).

5. Deduce from formula (5) that

$$J_{2n}(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos 2n\theta \cos (x \sin \theta)\, d\theta \qquad (n = 0, 1, 2, \qquad).$$

6. Deduce from formula (6) that

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin (2n - \quad \sin (x \sin \theta)\, d\theta \qquad (n = 1, 2,$$

7. Write the integral in Prob. 5 as the sum of the integrals over the intervals $(0, \pi/2 - \eta)$ and $(\pi/2 - \eta, \pi/2)$, where $\eta > 0$, and thus show that

$$\frac{\pi}{2}-\eta \quad \frac{\cos 2n\theta}{\cos \theta} \cos (x \sin \theta)\, d(x \sin$$

By integration by parts, show that the absolute value of the integral appearing here is not greater than a positive number M_η, independent of x. Hence, given any small positive number ϵ, by first selecting η sufficiently small and then x large, show that

$$< \epsilon \qquad \text{when } x >$$

This establishes formula (8) when $n = 0, 2, 4, \cdots$, there.

8. Apply the procedure of Prob. 7 to the formula in Prob. 6, and thus complete the proof of formula (8).

9. Note that the functions $\cos (x \sin \theta)$ and $\sin (x \sin \theta)$, of the variable θ, satisfy the conditions in our theorem in Sec. 38; also, since they are even and odd functions, respectively, the series of absolute values of their Fourier coefficients converges. Deduce that the series

is absolutely convergent for every x.

64. The Zeros of $J_n(x)$. The following theorem gives further information of importance in the applications of Bessel functions to boundary value problems.

Theorem 1. *For any given real n the equation $J_n(x) = 0$ has an infinite number of real positive roots $x_1, x_2, \cdots, x_m, \cdots,$ which become infinite with m.*

This will first be proved when $-\frac{1}{2} < n \leqq \frac{1}{2}$. The proof for every real n will then follow from Rolle's theorem. For if $J_n(x)$ vanishes when $x = x_1$ and $x = x_2$, for any real n, then so do $x^n J_n(x)$ and $x^{-n} J_n(x)$, and hence their derivatives vanish at least once between x_1 and x_2. But it was shown in Sec. 62 that these derivatives are $x^n J_{n-1}(x)$ and $-x^{-n} J_{n+1}(x)$, respectively.

Therefore between two zeros of $J_n(x)$ there is at least one zero of $J_{n-1}(x)$, and one of $J_{n+1}(x)$. So if there is an infinite number of zeros of $J_n(x)$ when $-\frac{1}{2} < n \leqq \frac{1}{2}$, the same is true when n is diminished or increased by unity, and repetitions of the argument show the same for any real n.

For the proof when $-\frac{1}{2} < n \leqq \frac{1}{2}$, we shall use the Lommel formula derived in the last section; namely,

$$(1) \qquad J_n(x) = \qquad \sin^{2n}\theta \cos(x\cos\theta)\, d\theta.$$

Now suppose that x is confined to the alternate intervals of length $\pi/2$ on the positive axis; that is,

$$x = m\pi + \frac{\pi}{2}t,$$

where $m = 0, 1, 2, \cdots,$ and $0 \leqq t \leqq 1$. Also let a new variable of integration λ, where

be introduced into the integral in formula (1). Then the integral becomes

$$\pi \int^{\frac{2x}{\pi}} \frac{\cos(\pi\lambda/2)\, d\lambda}{\qquad} .$$

and except for a factor which is always positive, this can be written

$$(2) \qquad \int_0^{2m+t} \frac{\cos(\pi\lambda/2)\, d\lambda}{[(2m+t)^2 - \lambda^2]^{-n+}}$$

The sign of $J_n(x)$ is therefore the same as the sign of integral (2). That integral can be broken up into this sum of integrals:

$$(3) \qquad -I_1 + I_2 - I_3 + \cdots + (-1)^m I_m + (-1)^m H_m,$$

where

$$I_j = (-1)^j \int_{2j-2}^{2j} \frac{\cos(\pi\lambda/2)\,d\lambda}{[(2m+t)^2 - \lambda^2]^{-n+\frac{1}{2}}} \qquad (j = 1, 2, \cdots, m),$$

$$\quad {}^{+t} \quad \frac{\cos(\pi\lambda/2)\,d\lambda}{[(2m+t)^2 - \lambda^2]^{-n+\frac{1}{2}}}.$$

Now let I_j be broken up into the sum of two integrals, one over the interval $(2j - 2, 2j - 1)$ and the other over the interval $(2j - 1, 2j)$. By substituting a new variable of integration μ into these integrals, where

$$\lambda = 2j - 1 - \mu$$

in the first, and

$$\lambda = 2j - 1 + \mu$$

in the second, it will be found that

$$\int_0^1 \qquad \sin\frac{\pi\mu}{2}$$

where

$$F_j(\mu) = [(2m \qquad\qquad + \mu)^2]$$
$$- [(2m + t)^2 - (2j - 1 -$$

Since $n - \frac{1}{2} \leqq 0$, the function $F_j(\mu)$ is never negative. By letting j assume continuous values and differentiating $F_j(\mu)$ with respect to j, we find that this function always increases in value with j.

The integral I_j is therefore a positive increasing function of j; that is,

$$0 \leqq I_1 \leqq I_2 \leqq \cdots \leqq I_m.$$

Furthermore H_m is not negative; because the numerator

$$\cos(\pi\lambda/2)$$

in the integral there can be written as $(-1)^m \cos(\pi\mu/2)$ when $\lambda = 2m + \mu$, and $\cos(\pi\mu/2)$ is positive.

Now the sum (3) can be written

$$(-1)^m[H_m + (I_m - I_{m-1}) + (I_{m-2} - I_{m-3}) + \cdots],$$

where the final term in the brackets is $(I_2 - I_1)$ if m is even, and I_1 if m is odd. The quantity in the brackets is therefore positive, and consequently the sign of $J_n(m\pi + \pi t/2)$ is that of $(-1)^m$; that is

$$J_n\left(m\pi + \frac{\pi}{2}t\right) > 0 \quad \text{if } m = 0, 2, 4,$$
$$< 0 \quad \text{if } m = 1, 3, 5,$$

Since $J_n(x)$ is a continuous function of x, its graph therefore crosses the x-axis between $x = \pi/2$ and $x = \pi$, and again between $x = 3\pi/2$ and $x = 2\pi$, and so on, when $-\frac{1}{2} < n \leqq \frac{1}{2}$. That is, $J_n(x)$ vanishes at an infinite number of points $x = x_1, x_2, \cdots$, where x_m tends to infinity with m. The theorem is therefore proved.

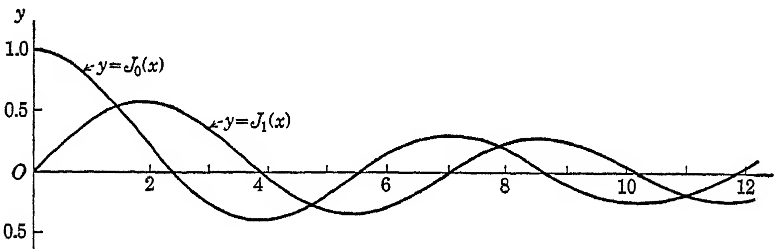

FIG. 9.

It follows at once from Rolle's theorem that *the equation*

$$J'_n(x) = 0$$

also has an infinite number of positive roots x'_m $(m = 1, 2, \cdots)$, and x'_m tends to infinity with m.

It should be observed that whenever x_m is a zero of $J_n(x)$, the number $-x_m$ is also a zero. This is true for any n, as is evident from our series for $J_n(x)$, Sec. 60.

The difference between successive roots of $J_n(x) = 0$ can be shown to approach π as the roots become larger.

Tables of numerical values of $J_n(x)$, and of the zeros of these functions, will be found in the references at the end of this chapter.* We list below the values, correct to four significant

* See Refs. 1, 3, and 4.

figures, of the first five zeros of $J_0(x)$, and the corresponding values of $J_1(x)$.

$$J_0(x_m) = 0$$

m	1	2	3	4	5
x_m	2.405	5.520	8.654	11.79	14.93
$J_1(x_m)$	0.5191	−0.3403	0.2715	−0.2325	0.2065

The graphs of the functions $J_0(x)$ and $J_1(x)$ are shown in Fig. 9.

PROBLEMS

1. Draw the graph of $J_{\frac{1}{2}}(x)$. (See Prob. 2, Sec. 61.)
2. Draw the graph of $J_{-\frac{1}{2}}(x)$. (See Prob. 1, Sec. 61.)
3. Draw the graph of $J_2(x)$ by composition of ordinates, using recursion formula (6), Sec. 62, and the graphs of $J_0(x)$ and $J_1(x)$.

65. The Orthogonality of Bessel Functions. Since $J_n(r)$ satisfies Bessel's equation, we can write

$$r^2 J_n''(r) + rJ_n'(r) + (r^2 - n^2 \qquad = 0.$$

Substituting the new variable x, where $r = \lambda x$ and λ is a constant, it follows that

$$= 0;$$

that is, $J_n(\lambda x)$ satisfies Bessel's equation in the form

For each fixed n this form is a special case of the Sturm-Liouville equation

$$\frac{d}{dx}\left[r(x)\frac{dX}{dx}\right] \qquad = 0,$$

with the parameter written as λ^2 instead of λ (Sec. 24). The function $r(x) = x$ here; hence it vanishes when $x = 0$. It follows from Theorem 3, Sec. 25, that those solutions of equation (1) in an interval $0 < x < c$, which satisfy the boundary condition

$$J_n(\lambda c) = 0,$$

form an orthogonal set of functions on that interval, with respect to the weight function $p(x) = x$. It will be observed case $q(x) = -n^2/x$, so it is discontinuous when $n = 0$; but this does not affect the proof of the

solution of equation (1), and it was shown in the last $J_n(\lambda c) = 0$ if $\lambda c = x_j$ $(j = 1, 2, \cdots)$, where x_j e zeros, infinite in number, of $J_n(x)$. λ_j denote the corr values of λ, so that

Then the functions $J_n(\lambda_j x)$ $(j = 1,$), are orthogonal derivatives $J_n'($ This is bly at $x = 0$, an vs that $xJ_n'(\lambda_j x)$

follows:

Theorem 2. *Let* λ_j $(j = 1,$ *the positive roots of the*

$$J_n(\lambda c) = 0, \tag{2}$$

where n is fixed and $n \geqq 0$. Then the functions $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$, $\cdots$ form an orthogonal set in the interval $(0, c)$ with respect to the weight function x; that is,

$$\int_0^c xJ_n(\lambda_j x)J_n(\lambda_k x)\,dx = 0 \qquad \text{if } k \neq j. \tag{3}$$

No new functions of the set are obtained by using the negative roots of equation (2), for an inspection of the series for J_n shows that

$$J_n(-\lambda_j x) = (-1)^n J_n(\lambda_j x).$$

It should be carefully noted that n is the same for all functions of the set; *hence an infinite number of sets has been determined here,* one for each n $(n \geqq 0)$.

Another boundary condition at $x = c$ which gives still other orthogonal systems can be seen by examining the proof of the above theorem. For *any* two distinct real values λ_j and λ_k of the parameter λ, the functions $J_n(\lambda_j x)$ and $J_n(\lambda_k x)$ satisfy

equation (1); hence

$$\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda_k x)\right] + \left(\lambda_k^2 x - \frac{n^2}{x}\right)J_n(\lambda_k x) = 0.$$

Multiplying the first of these equations by $J_n(\lambda_k x)$ and the second by $J_n(\lambda_j x)$, then subtracting and integrating, we find that

$$(\lambda_j^2 - \lambda_k^2)\int_0^c xJ_n(\lambda_j x)J_n(\lambda_k x)\,dx$$

$$\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda_k x)\right] - J_n(\lambda_k x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda_j x)\right]\Bigg\}\,dx$$

$$= \int_0^c \frac{d}{dx} \qquad d \quad J_n(\lambda_k x) - xJ_n(\lambda_k x) \quad d$$

When $n \geqq 0$, both terms in the brackets in the last expression vanish when $x = 0$; hence

(4) $$(\lambda_j^2 - \lambda_k^2)\int_0^c xJ_n(\lambda_j x)J_n(\lambda_k x)\,dx$$

where $J_n'(\lambda c)$ denotes the value of $(d/dr)J_n(r)$ when $r = \lambda c$.

Since $\lambda_j^2 - \lambda_k^2 \neq 0$, the orthogonality (3) exists whenever the right-hand member of equation (4) vanishes. This will be the case when λ_j and λ_k are two distinct values of λ which satisfy the equation

(5) $$\lambda c J_n'(\lambda c) = -hJ_n(\lambda c),$$

where h is any constant, including zero. The result can be written thus:

Theorem 3. *For any fixed n ($n \geqq 0$), the functions $J_n(\lambda_j x)$ ($j = 1, 2, \cdots$) form an orthogonal set in the interval $(0, c)$ with respect to the weight function x, when λ_j are the non-negative roots of equation* (5).

Here again, for every root λ_j there is a root $-\lambda_j$. This can be seen by writing equation (5) in the form

(6) $$(n + h)J_n(\lambda c) - \lambda c J_{n+1}(\lambda c) = 0.$$

Consequently the negative roots introduce no new characteristic functions. The details here can be left to the problems.

If $n + h \geqq 0$, equation (6) has no purely imaginary roots. This is easily seen by examining our series for $J_n(x)$. From now

on let us assume that $h \geqq 0$, as is usually the case in the applications; then $n + h \geqq 0$.

Similarly, equation (2) has no purely imaginary roots.

Now equation (5) can be written

$$(7) \qquad c \frac{d}{dx}[J_n(\lambda x)] + hJ_n(\lambda x) = 0 \qquad \text{when } x = c \quad (h \geqq 0);$$

hence it is a boundary condition of the type introduced earlier. It involves a linear combination of the dependent variable in Bessel's equation and the derivative of that variable.

Consider the Sturm-Liouville problem, consisting of Bessel's equation (1) and either one of the boundary conditions (2) or (7). A boundary condition at $x = 0$ is not involved because the function $r(x)$ in the general Sturm-Liouville equation is the independent variable x in this case, and it vanishes at $x = 0$. The characteristic functions here, $J_n(\lambda_j x)$, are continuous in the interval $(0, c)$, since $n \geqq 0$. Likewise for their first ordered derivatives, except possibly at the point $x = 0$; but the product $x(d/dx)\, J_n(\lambda_j x)$ is continuous and vanishes at the point $x = 0$, which is all that matters. Finally, note that the function $p(x)$ is also x itself here, and therefore it does not change sign in the interval $(0, c)$. Hence according to Theorem 4, Sec. 25, the characteristic numbers λ_j^2 are all real.

According to equation (6), $\lambda = 0$ is a root of equation (5) only if either $J_n(0) = 0$ or $n + h = 0$. In the first case the characteristic function $J_n(\lambda x)$ vanishes, so that the root $\lambda = 0$ can contribute a characteristic function only if $n = h = 0$. A root $\lambda = 0$ of equation (2) can never contribute a characteristic function.

We state our results as follows:

Theorem 4. *When $n \geqq 0$ and $h \geqq 0$, equations (2) and (5) have only real roots λ_j. For either equation we use only the non-negative roots, since no new characteristic functions correspond to the negative roots. The root $\lambda = 0$ is used only in the case of equation (5) with $n = h = 0$.*

PROBLEMS

1. Derive form (6) of equation (5).

2. Prove that when λ_j is any root of equation (6), $-\lambda_j$ is also a root of that equation.

66. The Orthonormal Functions. It can be shown that, when $n \geqq 0$, the function $J_n(x)$ is, except for a constant factor, the only solution of Bessel's equation that is bounded at $x = 0$. Hence it follows from the results of the last section that the functions $J_n(\lambda_j x)$ $(j = 1, 2, \cdots)$ represent all the characteristic functions of the Sturm-Liouville problem involved there, on the interval $(0, c)$. We can therefore anticipate an expansion of an arbitrary function in series of the functions of this set.

It should be observed that the orthogonality here with respect to the weight function x is the same as the ordinary orthogonality of the set of functions

Let us now find the value of the norm,

$$[J_n(\lambda_j x)]^2\, dx,$$

of the functions $J_n(\lambda_j x)$; these functions can then be normalized by multiplying them by the factors $(N_{nj})^{-\frac{1}{2}}$.

If we multiply the terms in Bessel's equation,

by the factor $(2x d/dx)J_n(\lambda x)$, we can write the equation as

$$\Bigg[$$

Integrating, and using integration by parts in the second term, we find that

$$- 2\lambda^2 \int_0^c x[J_n(\lambda x)]^2\, dx = 0.$$

Since

$$rJ_n'(r) = nJ_n(r) -$$

it follows that

$$2\lambda^2 \int_0^c x[J_n(\lambda x)]^2\, dx$$

or, since $n \geqq 0$,

$$(1) \quad \int_0^c x[J_n(\lambda x)]^2\,dx = \frac{c^2}{2}\{[J_n(\lambda c)]^2 + [J_{n+1}(\lambda c)]^2\}$$

Hence, *when* λ_j *represents the roots of the equation* $J_n(\lambda c) = 0$,

$$(2) \quad \int_0^c x[J_n(\lambda_j x)]^2\,dx = \frac{c^2}{2}[J_{n+1}(\lambda_j c)]^2 \qquad (j = 1, 2, \cdots).$$

When λ_j *represents the roots of equation* (5), *Sec.* 65, we have seen that

$$) = (n -$$

and hence formula (1) reduces to

$$(3) \quad \int_0^c x[J_n(\lambda_j x)]^2\,dx = \frac{\lambda_j^2 c^2 + h^2 - n^2}{2\lambda_j^2}[J_n(\lambda_j c)]^2$$

$$(j = 1, 2,$$

The normalized functions $\varphi_{nj}(x)$ can now be written

$$= \frac{J_n(\lambda_j x)}{/\ \overline{\lambda T}} \qquad (j = 1, 2,$$

where the norms N_{nj} are given for the two types of boundary conditions by equations (2) and (3). The set of functions $\{\varphi_{nj}(x)\}$ is *orthonormal* in the interval $(0, c)$ with x as a weight function; that is, for each fixed n $(n \geqq 0)$,

$$x\varphi_{nj}(x)\varphi_{nk}(x)\,dx = 0 \quad \text{if } j \neq k,$$

$$= 1 \quad \text{if } j = k \qquad (j, k = 1, 2, \cdots).$$

67. Fourier-Bessel Expansions of Functions. Let c_{nj} be the Fourier constants of a function $f(x)$ with respect to the functions $\varphi_{nj}(x)$ of our orthonormal set, where $f(x)$ is defined in the interval $(0, c)$. Then

$$c_{nj} = \int_0^c x\varphi_{nj}(x)f(x)\,dx$$

$$\int_0^c xJ_n(\lambda_j x)f(x)\,dx \qquad (j = 1, 2, \cdots),$$

and the generalized Fourier series corresponding to $f(x)$ can be written here as

$\geqq 0$, and $0 < x < c$.
of the formulas given in the last section for N_{nj}, this series can be written

$$\sum_{j=1}^{\infty} A_j J_n(\lambda_j x) \qquad (n \geqq 0), \tag{1}$$

where the coefficients A_j are defined as follows:

$$A_j = \frac{2}{c^2[J_{n+1}(\lambda_j c)]^2} \int_0^c x J_n(\lambda_j x) f(x)\, dx, \tag{2}$$

when $\lambda_1, \lambda_2, \cdots$ are the positive roots, in ascending order of magnitude, of the equation

$$J_n(\lambda c) = 0; \tag{3}$$

but

$$A_j = \frac{\lambda_j^2}{(\lambda_j^2 c^2 + \quad - n^2)[J_n(\lambda_j c)]^2} \int_0^c x J_n(\lambda_j \tag{4}$$

when $\lambda_1, \lambda_2, \cdots$ are the positive roots of the equation

$$\vdash hJ_n(\lambda c) = 0 \qquad (h \geqq 0,$$

$$A_1 = \tag{6}$$

It can be shown that, when $0 < x < c$, the series here does converge to $f(x)$ under the conditions given earlier for the representation of this function by its Fourier series. Let us state one such theorem here explicitly, and accept it without proof for the purposes of the present volume.*

Theorem 5. *Let $f(x)$ be any function defined in the interval $(0, c)$, such that $\int_0^c \sqrt{x}\, f(x)\, dx$ is absolutely convergent. Then at each point x $(0 < x < c)$ which is interior to an interval in which*

* A proof, using contour integrals in the complex plane, will be found in Ref. 1.

$f(x)$ is of bounded variation, series (1) *converges to*

that is,

(7) $\quad \frac{1}{2}[f(x+0)+f(x-0)] = \qquad\qquad (0 < x < c),$

he coefficients A_j are defined by equation (2) *or* (4), *and* $n \geqq 0,\ h \geqq 0$.

The theorem holds true for the special case $h = n = 0$, mentioned above, if A_1 is defined by formula (6).

It can be shown that all conditions here on $f(x)$ are satisfied everywhere, so that formula (7) *is true for every x* $(0 < x < c)$ *when $f(x)$ and its derivative $f'(x)$ are sectionally continuous in the interval* $(0, c)$. These conditions are narrower, but perhaps more practical for us, than those stated in the theorem.

Expansion (7) is usually called the *Fourier-Bessel expansion;* but when λ_j represents the roots of equation (5), the expansion is sometimes referred to as *Dini's.*

Other expansion formulas in terms of the Bessel functions J_n are known. There is, for instance, an integral representation of an arbitrary function which corresponds to the Fourier integral representation.

Suppose the interval $(0, c)$ is replaced by some interval (a, b) in the Sturm-Liouville problem with Bessel's equation, where (a, b) does not contain the point $x = 0$. Then a boundary condition is required at each end point $x = a$ and $x = b$, and the problem is no longer a singular case, but an ordinary special case, of the Sturm-Liouville problem. Hence the expansion in this case will be another one in series of Bessel functions; but here the functions of the second kind may be involved together with the functions J_n.

PROBLEMS

1. Expand the function $f(x) = 1$, when $0 < x < c$, in series of the functions $J_0(\lambda_j x)$, where λ_j are the positive roots of the equation

$J_0(\lambda c) = 0. \qquad$ *Ans.* $\quad 1 = \frac{2}{\ } \qquad\qquad (0 < x < c).$

2. In the expansion of $f(x) = 1$ $(0 < x < c)$ in series of $J_0(\lambda_j c)$, where λ_j are the non-negative roots of $J_0'(\lambda c) = 0$, show that $A_j = 0$ when $j = 2, 3, \cdots$, and $A_1 = 1$.

3. Expand the function $f(x) = 1$ when $0 < x < 1$, $f(x) = 0$ when $1 < x < 2$, $f(1) = \frac{1}{2}$, in series of $J_0(\lambda_j x)$ where λ_j are the roots of $J_0(2\lambda) = 0$. *Ans.* $f(x) = \frac{1}{2}\sum_{j=1}$ $(0 < x < 2)$.

4. Expand $f(x) = x$ $(0 < x < 1)$ in series of $J_1(\lambda_j x)$, where λ_j are the positive roots of $J_1(\lambda) = 0$. Also note the function represented by the series in the interval $-1 < x \leqq 0$.

Ans. $J_1(\lambda_j x)/[\lambda_j J_2(\lambda_j)]$ $(-1 < x < 1)$.

68. Temperatures in an Infinite Cylinder. Let the convex surface $r = c$ of an infinitely long solid cylinder, or a finite cylinder with insulated bases, be kept at temperature zero. If the initial temperature is a function $f(r)$, of distance from the axis only, the temperature at any time t will be a function $u(r, t)$. This function is to be found.

The heat equation in cylindrical coordinates, and the boundary conditions, are

$$u(c - 0, t) = 0 \qquad (t > 0), \tag{2}$$

$$u(r, +0) = f(r) \qquad (0 < r < c). \tag{3}$$

It will be supposed that $f(r)$ and $f'(r)$ are sectionally continuous in the interval $(0, c)$ and, for convenience, that $f(r)$ is defined to have the value $\frac{1}{2}[f(r + 0) + f(r - 0)]$ at each point r where it is discontinuous.

Particular solutions of equation (1) can be found by separation of variables. The function $u = R(r)T(t)$ is a solution, provided

$$RT' =$$

that is, if

Since the member on the left is a function of t alone, and that on the right is a function of r alone, they must be equal to a constant; say, $-\lambda^2$. Hence we have the equations

$$rR'' + R' + \lambda^2 rR = 0,$$
$$T' + k\lambda^2 T = 0.$$

The equation in R here is Bessel's equation (Sec. 65), in which $n = 0$. If the function RT is to satisfy the condition (2), then $R(r)$ must satisfy the condition

$$R(c) = 0.$$

According to Theorem 4, there are only real values of the parameter λ in Bessel's equation for which a solution exists and satisfies this condition, and the positive values alone yield all possible solutions. We are supposing that the function $R(r)$ and its derivative of the first order are continuous functions when $0 \leqq r \leqq c$. It can be shown that the second fundamental solution, $R = Y_0(\lambda r)$, of Bessel's equation, or Bessel's function of the second kind, is infinite when $r = 0$. Therefore the only functions $R(r)$ which satisfy the required conditions are $J_0(\lambda_j r)$, where λ_j are the positive roots of the equation

$$J_0(\lambda c) = 0. \tag{4}$$

The only particular solutions $u = RT$ of the heat equation (1) which satisfy the homogeneous boundary condition (2) are therefore (except for a constant factor)

$$u = J_0(\lambda_j r)e^{-k\lambda_j^2 t},$$

where λ_j are the positive roots of equation (4).

A series of these solutions,

$$u(r, t) = \sum_{j=1}^{\infty} A_j J_0(\lambda_j r)e^{-k\lambda_j^2 t}, \tag{5}$$

will formally satisfy the heat equation (1) and the condition (2); it will also satisfy the initial condition (3) provided the coefficients A_j can be determined so that

$$f(r) = \sum_{j=1}^{\infty} A_j J_0(\lambda_j r) \qquad (0 < r < c).$$

This is true, according to the Fourier-Bessel expansion, if

$$A_j = \frac{2}{c^2[J_1(\lambda_j c)]^2} \int_0^c rf(r)J_0(\lambda_j r)\, dr \qquad (j = 1, 2, \cdots). \tag{6}$$

The formal solution of the boundary value problem is therefore represented by series (5) with coefficients (6), where λ_j are the positive roots of equation (4). That is, our solution can

be written

$$\frac{)}{]^2} e^{-k\lambda_j^2 t} \int_0^c$$

This result can be fully *established as a solution* of the boundary value problem stated here, by following the method used in Sec. 46. For it can be shown that the numbers $1/[\lambda_j J_1^2(\lambda_j c)]$ are bounded for all the roots λ_j.* Consequently the numbers A_j/λ_j are bounded for all j $(j = 1, 2, \cdots)$, because $f(r)$ and $J_0(\lambda_j r)$ are bounded. Hence for each positive number t_0, the absolute values of the terms of series (5) are less than the constant terms

$$M\lambda_j \exp(-k\lambda_j^2 t_0)$$

for all r $(0 \leqq r \leqq c)$ and all t $(t \geqq t_0)$, where M is a constant. The series of these constant terms converges, since $\lambda_{j+1} - \lambda_j$ approaches π as j increases.

Series (5) therefore converges uniformly when $t > 0$, and so the function $u(r, t)$ represented by it is continuous with respect to r when $r = c$. But $u(c, t)$ is clearly zero; hence condition (2) is satisfied.

Since the derivatives of $J_0(\lambda_j r)$ are also bounded, it follows in just the same way that the differentiated series converge uniformly when $t > 0$, and hence that result (5) satisfies the heat equation (1).

Finally, owing to the convergence of series (5) when $t = 0$, Abel's test applies to show that $u(r, +0) = u(r, 0)$, when $0 < r < c$; hence the condition (3) is satisfied.

To determine conditions under which our solution is unique, we should need information about the uniform convergence of the Fourier-Bessel expansion. This matter is beyond the scope of our introductory treatment.

PROBLEMS

1. Write the solution of the above problem when the initial temperature $f(r)$ is a constant A, and $c = 1$. Give the approximate numerical values of the first few coefficients in the series.

Ans. $u = 2A[0.80J_0(2\ 4r)e^{-5.8kt} - .53J_0(5.5r)e^{-30kt}$

]

* This can be seen, for instance, from the asymptotic formulas for λ_j and $_1(x)$ developed in Ref. 1.

2. Over a long solid cylinder of radius 1 at temperature A throughout is tightly fitted a long hollow cylinder of the same material, with thickness 1 and temperature B throughout. The outer surface of the latter is then kept at temperature B. Find the temperatures in the composite cylinder of radius 2 so formed. This is a heat problem in shrunk fittings. (Note that it becomes a case of the problem in this section when B is subtracted from all temperatures.)

Ans. $u(r, t) = B +$ where λ_1, $\lambda_2, \cdots$ are the positive roots of $J_0(2\lambda) = 0$.

3. Derive the formula for the potential in a cylindrical space bounded by the surfaces $r = c$, $z = 0$, and $z = b$, when the first two surfaces are kept at potential zero and the third at potential $V = f(r)$.

Ans. $V(r, z) = \sum_{j=1}^{\infty} A_j J_0(\lambda_j r)(\sinh \lambda_j z / \sinh \lambda_j b)$, where λ_j are the positive roots of equation (4), and the coefficients A_j are given by equation (6).

4. Derive the formula for the steady temperatures $u(r, z)$ in the solid cylinder bounded by the surfaces $r = 1$, $z = 0$, and $z = 1$, when the first surface is kept at temperature $u = 0$, the last at $u = 1$, and the surface $z = 0$ is insulated.

69. Radiation at the Surface of the Cylinder. Let the surface of the infinite cylinder of the last section, instead of being kept at temperature zero, undergo heat transfer into surroundings at temperature zero, according to Newton's law. The flux of heat through the surface $r = c$ is then proportional to the temperature of the surface; that is,

$$-K\frac{\partial u}{\partial r} = Eu \qquad \text{when } r = c,$$

where K is the conductivity of the material in the cylinder and E is the external conductivity, or emissivity. Let us write $h = cE/K$.

The boundary value problem for the temperature $u(r, t)$ can be written as follows:

$$\left.\frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad (0 < r < c,\ t > 0),$$

(2) $$\partial r \qquad = -hu(c - 0, t) \qquad (t > 0),$$

(3) $$u(r, +0) = f(r) \qquad (0 < r < c).$$

The particular solution of equation (1) found before,

$$u =$$

will satisfy condition (2) provided λ is any root λ_j of the equation

$$c \frac{d}{dr} J_0(\lambda r) = -hJ_0(\lambda r) \qquad \text{when } r = c,$$

that is, of the equation

$$\lambda c J_0'(\lambda c) = -hJ_0(\lambda c). \tag{4}$$

Hence the solution of the problem (1)-(3) can be written

$$u(r, t) = \tag{5}$$

where λ_j are the positive roots of equation (4), and where, according to Theorem 5,

$$A_j = \frac{2\lambda_j^2}{(\lambda_j^2 c^2 + h^2)[J_0(\lambda_j c)]^2} \int_0^c r J_0(\lambda_j r) f(r)\, dr \quad (j = 1, 2, \cdots).$$

If $h = 0$, *then* $\lambda_1 = 0$, and the first term of the series in formula (5) is the constant A_1, where

This is the case if the surface $r = c$ is thermally insulated.

PROBLEMS

1. Find the steady temperatures $u(r, z)$ in a solid cylinder bounded by the surfaces $r = 1$, $z = 0$, and $z = L$ if the first surface is insulated, the second kept at temperature zero, and the last at temperature $f(r)$.

Ans. $u = \frac{2z}{L} \int_0^1 r' f(r')$

$j=2$

where $\lambda_2, \lambda_3, \cdots$ are the positive roots of $J_1(\lambda) = 0$.

2. Find the steady temperatures in a semi-infinite cylinder bounded by the surfaces $r = 1$ $(z \geqq 0)$ and $z = 0$, if there is surface transfer of heat at $r = 1$ into surroundings at temperature zero, and the base $z = 0$ is kept at temperature $u = 1$.

3. Show that the answer to Prob. 1 reduces to $u = Az/L$ when $f(r)$ is a constant A.

70. The Vibration of a Circular Membrane. A membrane, stretched over a fixed circular frame $r = c$ in the plane $z = 0$, is given an initial displacement $z = f(r, \varphi)$ and released from rest in that position. Its displacement $z(r, \varphi, t)$, where (r, φ, z) are cylindrical coordinates, will be found as the continuous solution of the following boundary value problem:

(2) $= 0,$

(3) $0) = f(r,$

The function $z = R(r)\Phi(\varphi)T(t)$ satisfies equation (1) if

T'' $1\ \Phi''$

where $-\lambda^2$ is any constant, according to the usual argument. Hence $T = \cos(a\lambda t)$ if the first of conditions (3) is to be satisfied. Also R and Φ must satisfy the equations

Φ''

$\overline{R}$

where μ^2 is any constant, since the member on the left cannot vary with either φ or r.

Hence

$$\Phi = A \cos \mu\varphi \quad \sin$$

But z must be a periodic function of φ with period 2π; hence $\mu = n$ $(n = 0, 1, 2, \cdots)$. The equation in R then becomes Bessel's equation with the parameter λ,

$$r^2R'' + rR' + (\lambda^2r^2 - n^2)R = 0,$$

and so $R = J_n(\lambda r)$. The solution $z = R\Phi T$ will satisfy condition (2) if λ is any of the roots λ_{nj} of the equation

(4) $$J_n(\lambda c) = 0 \qquad (n = 0, 1, 2, \cdots).$$

Therefore if A_{nj} and B_{nj} are constants, the functions

$$J_n(\lambda_{nj}r)(A_{nj}\cos n\varphi + B_{nj}\sin n\varphi)\cos(a\lambda_{nj}t)$$

are solutions of (1) which satisfy all but the last of conditions (3). The function

$$(5)\quad z(r, \varphi, t) \qquad \sum_{n=0}\sum_{j} \quad {}_{nj} \cos n\varphi + B_{nj} \sin n\varphi) \cos (a\lambda_{nj}t)$$

satisfies this last condition also, provided the coefficients are such that

$$(6)\quad f(r, \varphi) \qquad \sum^{\infty} A_{nj}J_n(\lambda_{nj}r)\Big] \cos n\varphi \quad \sum_{j=1}$$

when $-\pi < \varphi \leqq \pi$, $0 \leqq r \leqq c$.

For each fixed r, the right-hand member of equation (6) is the Fourier series for $f(r, \varphi)$, in the interval $-\pi < \varphi < \pi$, provided the coefficients of $\cos n\varphi$ and $\sin n\varphi$ are the Fourier coefficients; that is, if

$$(7)\quad \sum_{j=1}^{\infty} A_{nj}J_n(\lambda_{nj}r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r, \varphi) \cos n\varphi \, d\varphi \quad (n = 1, 2, \cdots),$$

$$d\varphi \qquad (n = 0),$$

$$(8)\quad \sum B_{nj}J_n(\lambda_{nj}r) = \frac{1}{\pi}\int^{\pi} f(r, \varphi) \sin n\varphi \, d\varphi \quad (n = 1, 2, \cdots).$$

But the left-hand member of equation (7) is a series of Bessel functions which must represent the function of r on the right when $0 \leqq r \leqq c$. It is the Fourier-Bessel expansion of that function, provided

$$(9)\quad A_{nj} = \frac{2}{\pi c^2[J_{n+1}(\lambda_{nj}c)]^2}\int_0^c rJ_n(\lambda_{nj}r)\, dr \int_{-\pi}^{\pi} f(r, \varphi) \cos n\varphi \, d\varphi$$
$$(n = 1, 2, \cdots),$$

Similarly, according to equation (8),

$$(11)\quad B_{nj} = \frac{2}{\pi c^2[J_{n+1}(\lambda_{nj}c)]^2}\int_0^c rJ_n(\lambda_{nj}r)\, dr \int_{-\pi}^{\pi} f(r, \varphi) \sin n\varphi \, d\varphi.$$

The displacement is therefore given by formula (5) when the coefficients have the values given by equations (9), (10), and (11), and λ_{nj} are the positive roots of equation (4), provided series (5) has the necessary properties of convergence, differentiability, and continuity.

PROBLEMS

1. Derive the formula for the displacement of the above membrane if the initial displacement is $f(r)$, a function of r only. Also show that when $f(r) = AJ_0(\lambda_k r)$, where λ_k is a root of $J_0(\lambda c) = 0$, the displacement of the membrane is periodic in time, so that the membrane gives a musical note.

Ans. $z(r, t) = \frac{2}{c^2} \sum_{j=1}$

are the positive roots of $J_0(\lambda c) = 0$.

2. Find the displacements in the above membrane if at $t = 0$ every point within the boundary of the membrane has the velocity $\partial z/\partial t = 1$ in the position $z = 0$. This is the case if the membrane and its frame are moving as a rigid body with unit velocity and the frame is suddenly brought to rest.

$$z(r, t) = \frac{2}{ac} \sum_{j=1}^{\infty} \frac{\sin a\lambda_j t}{\lambda_j^2 J_1(\lambda_j c)} J_0(\lambda_j r), \text{ where } \lambda_j \text{ are the positive}$$

roots of $J_0(\lambda c) = 0$.

3. Derive the following formula for the temperatures in a solid cylinder with insulated bases, if the initial temperature is $u = f(r, \varphi)$, and the surface $r = c$ is kept at temperature zero:

$$u(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} J_n(\lambda_{nj} r)(A_{nj} \cos n\varphi + B_{nj} \sin n\varphi) e^{-k\lambda_{nj}^2 t},$$

where A_{nj} and B_{nj} have the values given by equations (9), (10), and (11), and λ_{nj} are the positive roots of equation (4).

4. Derive the formula for the temperature $u(r, z, t)$ in a solid cylinder of radius c and altitude L whose entire surface is kept at temperature zero and whose initial temperature is A, a constant. Show that the formula can be written

$$u(r, z, t) = Au_1(z, t)u_2(r, t),$$

where $u_1(z, t)$ is the temperature in a slab whose faces $z = 0$ and $z = L$ are kept at temperature $u_1 = 0$, and whose initial temperature is $u_1 = 1$; while $u_2(r, t)$ is the temperature in an infinite cylinder whose surface

$r = c$ is kept at $u_2 = 0$, and whose initial temperature is $u_2 = 1$. That is,

$$m_n \qquad m_n \qquad (2n -$$

and

$$r,\ t) = \frac{}{c}$$

5. Derive the following formula for the temperatures in an infinitely long right-angled cylindrical wedge bounded by the surface $r = c$ and the planes $\varphi = 0$ and $\varphi = \pi/2$, when these three surfaces are kept at temperature zero and the initial temperature is $u = f(r, \varphi)$:

$$u(r, \varphi, t) = \sum_{n=1} \sum_{j=1} \qquad \sin$$

where λ_{nj} are the positive roots of $J_{2n}(\lambda c) = 0$, and A_{nj} are given by the formula

$$, \int_0^c \qquad dr.$$

6. If the planes of the wedge in Prob. 5 are $\varphi = 0$ and $\varphi = \varphi_0$, show that the formula for the temperature will in general involve Bessel functions of nonintegral orders. Derive the formula for $u(r, \varphi, t)$ in this case.

7. Solve Prob. 5 if the planes $\varphi = 0$ and $\varphi = \pi/2$ are insulated, instead of being kept at temperature zero. When $f(r, \varphi) = 1$, show that your formula reduces to

$$\frac{J_0(\lambda_j r)}{} e^{-k\lambda_j^2 t},$$

where λ_j are the positive roots of $J_0(\lambda c)$ 0; thus u is independent of the angle φ.

8. Solve Prob. 5 if all three surfaces $r = c$, $\varphi = 0$, and $\varphi = \pi/2$, are insulated instead of being kept at temperature zero.

9. Let $u(r, t)$ be the temperature in a thin circular plate whose edge, $r = 1$, is kept at temperature $u = 0$, and whose initial temperature is $u = 1$, when there is surface heat transfer from the circular faces to surroundings at temperature zero. The heat equation can then be written

$$\frac{\partial u}{\partial t} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} -$$

where h is a positive constant. Derive the following formula for $u(r, t)$:

$$\infty$$

where λ_j are the positive roots of $J_0(\lambda) = 0$.

10. Solve Prob. 9 if there is also surface heat transfer at the edge $r = 1$, instead of a fixed temperature there, so that

$$\overline{\partial r} \qquad \text{when } r =$$

11. Derive the following formula for the potential $V(r, z)$ in the semi-infinite cylindrical space, $r \leqq 1$, $z \geqq 0$, if the surface $r = 1$ is kept at potential $V = 0$, and the base $z = 0$ at $V = 1$:

$$V = 2 \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r)}{\lambda_j J_1(\lambda_j)} e^{-\lambda_j z},$$

where λ_j are the positive roots of $J_0(\lambda) = 0$.

12. Let $V(r, z)$ be the potential in the space inside the cylinder $r = c$, when the surface $r = c$ is kept at the potential $V = f(z)$, where the given function $f(z)$ is defined for all real z. Derive the following formula for $V(r, z)$:

where $i = \sqrt{-1}$.

13. Let $V(r, z)$ be the potential in the semi-infinite cylindrical space $r \leqq 1$, $z \geqq 0$, if $\partial V/\partial z = 0$ on the surface $z = 0$ and if, on the surface $r = 1$, $V = 1$ when $0 < z < 1$, while $V = 0$ when $z > 1$. Show that

$$\cdot \cos \alpha z \sin \alpha \, d\alpha.$$

REFERENCES

1. Watson, G. N.: "A Treatise on the Theory of Bessel Functions," 1922.
2. Bowman, F.: "Introduction to Bessel Functions," 1938.
3. Gray, A., G. B. Mathews, and T. M. MacRobert: "A Treatise on Bessel Functions and Their Applications to Physics," 1931.
4. Jahnke, E., and F. Emde: "Tables of Functions," 1933.

CHAPTER IX

LEGENDRE POLYNOMIALS AND APPLICATIONS

71. Derivation of the Legendre Polynomials. Any solution of the differential equation

$$(1) \qquad (1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0,$$

known as *Legendre's equation*, is called a *Legendre function.* Later on we shall see how this equation arises in the process of obtaining particular solutions of Laplace's equation in spherical coordinates, when x is written for $\cos\theta$. We shall consider here only the cases in which the parameter n is zero or a positive integer.

To find a solution which can be represented by a power series, if any such exist, we substitute

$$(2) \qquad y = \sum_{j=0}^{\infty} a_j x^j,$$

into equation (1) and determine the coefficients a_j. The substitution gives

$$\cdot\, n(n \qquad = 0$$

or

zero gives

$$= 0$$

$$j = 0, 1, 2,$$

which is a recursion formula giving each coefficient in terms of the second one preceding it, except for a_0 and a_1. It can be written

$$a_{j+2} = -\frac{(n-j)(n+j+1)}{(j+1)(j+2)} a_j \qquad (j = 0, 1, 2, \cdots). \tag{4}$$

The power series (2) is therefore a solution of Legendre's equation within its interval of convergence, provided its coefficients satisfy relation (4); this leaves a_0 and a_1 as arbitrary constants. But since n is an integer, it follows from relation (4) that $a_{n+2} = 0$, and consequently

$$a_{n+4} = a_{n+6} = \cdots = 0.$$

Also, when $a_0 = 0$, then $a_2 = a_4 = \cdots = 0$; and when $a_1 = 0$, then $a_3 = a_5 = \cdots = 0$.

Hence if n is odd and a_0 is taken as zero, the series reduces to a polynomial of degree n containing only odd powers of x. If n is even and a_1 is set equal to zero, the series reduces to a polynomial of degree n containing only even powers of x. So there is always a polynomial solution of equation (1), and for it no question of convergence arises.

These polynomials can be written explicitly in descending powers of x whether n is even or odd. All the nonvanishing coefficients can be written in terms of a_n by means of recursion formula (4); thus

$$[illegible]\ \frac{n(n-1)}{[illegible]} a_n,$$

$$[illegible] -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$[illegible]\ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} [illegible],$$

and so on. Hence the polynomial

$$y = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} \right. \tag{5}$$

$$\left. + \frac{[illegible]}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} - \cdots \right]$$

is a particular solution of Legendre's equation.

Here the coefficient a_n is arbitrary. It turns out to be convenient to give it the value

$$a_n = \frac{(2n-1)(2n-3) \qquad 3 \cdot 1}{n!} \qquad \text{if } n = 1, 2,$$

$$a_0 = 1.$$

With this choice of a_n functions (5) are known as the *Legendre polynomials:*

$$(6) \quad P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\left[x^n - \quad n(n - \right.$$

$$+ \frac{- 1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} - \cdot \cdot$$

The function $P_n(x)$ is a polynomial in x of degree n, containing only even powers of x if n is even and only odd powers if n is odd. It is therefore an even or odd function according as n is even or odd; that is,

The first few polynomials are as follows:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad = \tfrac{1}{2}(3x^2 - 1),$$

$$- 3x), \qquad 3),$$

PROBLEMS

1. Show that formula (6) for $P_n(x)$ can be written in the following compact form:

$$x^{n-2j}.$$

where $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd.

2. With the aid of the formula in Prob. 1, show that

$$P_{2n-1}(0) = 0, \qquad (n = 1, 2, \cdots).$$

72. Other Legendre Functions. When n is a positive integer or zero, we obtained the solution $y = P_n(x)$ of Legendre's equation by setting one of the two arbitrary constants a_0 or a_1 in the

series solution equal to zero. If these constants are left arbitrary, it is easily seen that the *general solution* of Legendre's equation can be written

$$y = AP_n(x) + BQ_n(x), \tag{1}$$

where A and B are arbitrary constants. The functions $Q_n(x)$ here, called *Legendre's functions of the second kind*, are defined by the following series when $|x| < 1$:

$$Q_n(x) \qquad \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 -$$

if n is even; and

$$= \alpha_0\left[1 - \frac{n(n}{2!} \qquad \frac{\cdot 3)}{} x^4 -\right.$$

if n is odd; and where

$$\frac{n}{2} \quad \frac{2 \cdot 4 \cdots n}{1 \cdot 3 \cdot 5 \cdots (n-1)}$$

$$\frac{n+1}{2} \quad \frac{2 \cdot 4 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}.$$

Of course $P_n(x)$ is a solution for all x. But when $|x| > 1$, the above series for $Q_n(x)$ do not converge. To obtain a second fundamental solution in that case, a series of descending powers of x is used. The following solution so obtained is taken as the definition of $Q_n(x)$ when $|x| > 1$:

$$Q_n(x) = \frac{n!}{\cdot 3 \cdot 5 \cdot} \qquad \frac{}{+1)} \qquad \frac{}{2(2n+3)} \qquad \frac{+4)}{} x^{-n-5} +$$

Both $P_n(x)$ and $Q_n(x)$ are special cases of the function known as the hypergeometric function.

When n is not an integer, the two fundamental solutions of Legendre's equation can be written as infinite series. These

are both power series when $|x| < 1$; but when $|x| > 1$, they are series in descending powers of x.

Of these various Legendre functions the polynomials $P_n(x)$ are by far the most important. Let us now continue with the study and application of those polynomials.

73. Generating Functions for $P_n(x)$. If $-1 \leqq x \leqq 1$, the function

$$(1 - 2xz + z^2)^{-\frac{1}{2}}$$

and its derivatives of all orders with respect to z exist when $|z| < 1$. For these functions are infinite only when

$$1 - 2xz + z^2 = 0,$$

that is, if

$$z = x \pm \sqrt{x^2 - 1} = \cos\theta + i \sin\theta,$$

where we have written $\cos\theta$ for x. But this shows that $|z| = 1$. It is shown in the theory of functions of complex variables that such regular functions of z are always represented by their Maclaurin series within the region of regularity ($|z| < 1$, in this case).

It will now be shown that the coefficients of the powers of z in that series representation of the above function are the Legendre polynomials; that is, when $-1 \leqq x \leqq 1$ and $|z| < 1$,

$$(1) \quad (1 - 2xz + z^2)^{-\frac{1}{2}} \\ = P_0(x) + P_1(x)z + P_2(x)z^2 + \cdots + P_n(x)z^n + \cdots .$$

To find the coefficients, it is best to write the expansion by means of the binomial series:

$$[1 - z(2x - z)]^{-\frac{1}{2}} = 1 + \frac{1}{2}z(2x - z) + \frac{1 \cdot 3}{2^2 \cdot 2!}z^2(2x - z)^2$$

The terms in z^n come from the term containing $z^n(2x - z)^n$ and preceding terms, so that the total coefficient of z^n is

$$\frac{1 \cdot 3 \cdots (2n-1)}{} (2x)^n - \frac{1 \cdot 3 \cdots (2n-3)}{2^{n-1}(n-1)!}\frac{(n-1)}{1!}(2x)^{n-2} \\ + \frac{1 \cdot 3 \cdots (2n-5)}{2^{n-2}(n-2)!}\frac{(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots .$$

This can be written

$$-\ 3) \qquad \Big]. $$
$$-\ 3)$$

But this is $P_n(x)$; hence relation (1) is established. Incidentally, this shows the reason for the value assigned to a_n in our definition of $P_n(x)$ (Sec. 71).

For $x = 1$, expansion (1) becomes

Consequently

$$P_n(1) = 1 \qquad (n = 0, 1, 2,$$

Likewise, putting $x = 0$ gives

$$1) \quad 2 \cdot 4 \cdot \cdot$$

and therefore

$$1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n - 1).$$

$$P_{2n-1}(0) = 0 \qquad (n = 1, 2, \cdot \cdot \cdot).$$

By differentiating equation (1) with respect to z and multiplying the resulting equation by $(1 - 2xz + z^2)$, the following identity in z is found:

$$(x - z)(1 - 2xz + z^2)^{-\frac{1}{2}} = (x - z) \sum_0^\infty P_n(x) z^n$$
$$= (1 - 2xz + z^{\text{:}}$$

Equating the coefficients of z^n in the last two expressions, it follows that

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (n = 1, 2, \cdot \cdot \cdot);$$

this is a *recursion formula* for $P_n(x)$. It is valid for all values of x.

The result of integrating polynomial (6), Sec. 71, n times from 0 to x is

$$\frac{(2n-1)(2n-3)\cdots 1}{} \,\Big|\, x^{2n} - nx^{2n-2}$$

$$+ \cdot \frac{}{2!} \cdot x^{2n-4} -$$

and the expression in brackets differs from $(x^2 - 1)^n$ only by a polynomial of degree less than n. By differentiating n times, then, it follows that

This is *Rodrigues' formula* for the Legendre polynomials.

PROBLEMS

1. Show that the derivatives of Legendre polynomials have the properties

$$P'_{2n}(0) = 0; \qquad P'_{2n+1}(0) = (-1)^n \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)}.$$

The latter can be found by differentiating equation (1) with respect to x and setting $x = 0$.

2. Carry out the details of the derivation of Rodrigues' formula.

3. Using Rodrigues' formula, show that

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x) \qquad (n = 1, 2, \cdots).$$

4. Using the formula in Prob. 3, obtain the integration formula

$$\int_x^1 P_n(x)\,dx = \frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)] \qquad (n = 1, 2, \cdots).$$

74. The Legendre Coefficients. When $-1 \leqq x \leqq 1$, we have just shown that $P_n(x)$ is the coefficient of z^n in the expansion of the generating function $(1 - 2xz + z^2)^{-\frac{1}{2}}$ in powers of z. When

$$x = \cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}),$$

this generating function can be written

$$[1 - z(e^{i\theta} + e^{-i\theta}) + z^2]^{-\frac{1}{2}} = (1 - ze^{i\theta})^{-\frac{1}{2}}(1 - ze^{-i\theta})^{-\frac{1}{2}}$$

and therefore as the product of the series

$$1 + \frac{1}{2}ze^{i\theta} + \frac{1 \cdot 3}{2 \cdot 4}z^2e^{2i\theta} + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}z^ne^{in\theta} + \cdots$$

and the series

$$(2n -$$

$$\overline{2 \cdot 4}$$

The coefficient of z^n in this product is

$$\overline{2 \cdot 4}$$

$$+ \frac{1}{2} \frac{2n}{2n - 1} (e^{i(n-2)\theta} + e^{-i(n-2)\theta}) + \cdots \Big];$$

hence this is P_n (cos). Thus we have the following formula for this function:

(1) $P_n(\cos \theta)$

$$\frac{1 \cdot 3 \cdots (2n -}{n!\,2^{n-1}} \qquad \cdot \cos (n -$$

$$+ \frac{1 \cdot 3 \quad n(n-1)}{1 \cdot 2(2n-1)(2n-3)} \cos (n -$$

where the final term T_n is the term containing $\cos \theta$ if n is odd; but it is half the constant term indicated if n is even.

These functions are called the *Legendre coefficients*. Tables of their numerical values will be found in some of the more extensive books of mathematical tables, or in Ref. 3 at the end of this chapter.

According to formula (1), the first few functions are

$$\begin{aligned}
P_0(\cos \theta) &= 1, \\
P_1(\cos \theta) &= \cos \theta, \\
P_2(\cos \theta) &= \tfrac{1}{4}(3 \cos 2\theta + 1), \\
P_3(\cos \theta) &= \tfrac{1}{8}(5 \cos 3\theta + 3 \cos \theta), \\
P_4(\cos \theta) &= \tfrac{1}{64}(35 \cos 4\theta + 20 \cos 2\theta + 9).
\end{aligned}$$

The coefficients of the cosines in formula (1) are all positive. Consequently $P_n(\cos \theta)$ has its greatest value when $\theta = 0$. Since $P_n(1) = 1$, it follows that $P_n(\cos \theta) \leqq 1$. Also, each cosine is greater than or equal to -1, so that $P_n(\cos \theta) \geqq -1$. That is, *the Legendre coefficients are uniformly bounded as follows:*

$$|P_n(\cos \theta)| \leqq 1 \qquad (n = 0, 1, 2, \cdots),$$

for all real values of θ.

75. The Orthogonality of $P_n(x)$. Norms. Legendre's equation can be written in the form

$$\frac{d}{dx} \tag{1}$$

It is clearly a special case of the Sturm-Liouville equation (Sec. 24), in which the parameter λ has been assigned the values

$$\lambda = n(n+1) \qquad (n = 0, 1, 2, \tag{2}$$

In this case $r(x) = 1 - x^2$, $q(x) = 0$, and the weight function $p(x) = 1$.

Since $r(x) = 0$ when $x = \pm 1$, no boundary conditions need accompany the differential equation to form the Sturm-Liouville problem on the interval $(-1, 1)$. It is only required that the characteristic functions and their first ordered derivatives be continuous when $-1 \leqq x \leqq 1$. But the polynomials $P_n(x)$ are solutions of equation (1), and, of course, they have these required continuity properties.

The Legendre polynomials $P_n(x)$ are therefore the characteristic functions of the Sturm-Liouville problem here, corresponding to the characteristic numbers (2). According to Sec. 25, then, *the functions $P_n(x)$ form an orthogonal set in the interval* $(-1, 1)$, *with respect to the weight function* $p(x) = 1$; *that is,*

$$P_m(x)P_n(x)\,dx = 0 \qquad (m, n = 0, 1, 2, \qquad). \tag{3}$$

Furthermore, there can be no characteristic functions of the Sturm-Liouville problem here which correspond to complex values of the parameter λ, because $p(x)$ does not change sign. We shall soon see that the functions $P_n(x)$ and the numbers (2) are the only possible characteristic functions and numbers of the problem.

To find the norm of $P_n(x)$, that is, the value of the integral in (3) when $m = n$, a simple method consists first of squaring both members of equation (1), Sec. 73, to obtain the formula

$$(1 - 2xz +$$

We now integrate both members here with respect to x over the interval $(-1, 1)$ and observe that the product terms on the right

vanish in view of the orthogonality property (3). Thus,

$$\int_{-1}^{1} \frac{}{1-} \qquad\qquad {}^{2n} \int_{-1}^{1} [P_n(\qquad\qquad 1).$$

The integral on the left has the value

$$-\frac{1}{2z}\log(1 - 2xz + z^2)\Big]_{-1}^{1}$$

$$-\ 1 \qquad 1 + z$$

By equating the coefficients of z^{2n} in the last two series, we have the following formula for *the norm of* $P_n(x)$:

(4) $$(n = 0, 1, 2, \qquad).$$

The orthonormal set of functions here in the interval $(-1, 1)$ *is therefore* $\{\varphi_n(x)\}$, *where*

$$\varphi_n(x) = \sqrt{n + \tfrac{1}{2}}\, P_n(x) \qquad (n = 0, 1, 2, \cdots).$$

Since $[P_n(x)]^2$ and the product $P_m(x)P_n(x)$, in which m and n are both even or both odd, are *even* functions of x, it follows from formulas (4) and (3) that the *polynomials of even degree,*

(5) $$\sqrt{2n + 1}\, P_n(x) \qquad (n = 0, 2, 4,$$

form an orthonormal set of functions in the interval $(0, 1)$; and the same is true for the polynomials of *odd degree,* represented by (5) when $n = 1, 3, 5, \cdots$.

PROBLEMS

1. Establish the orthogonality property (3) by using Rodrigues' formula for $P_n(x)$ and successive integration by parts.

2. State why it is true that

$$P_n(x)\, dx = 0 \qquad (n = 1, 2, 3, \cdots).$$

3. Use the method of Prob. 1 to obtain formula (4) for the norm of

76. The Functions $P_n(x)$ as a Complete Orthogonal Set. Let us now prove the following theorem:

Theorem 1. *In the interval* $(-1, 1)$ *the orthogonal set of functions consisting of all the Legendre polynomials*

$$P_n(x) \qquad (n = 0, 1, 2, \cdots)$$

is complete with respect to the class of all functions which, together with their derivatives of the first order, are sectionally continuous in $(-1, 1)$.

We are to prove that if $\psi(x)$ is a function of this class which is orthogonal to each of the functions $P_n(x)$, then $\psi(x) = 0$ except at a finite number of points in the interval.

Let us suppose, then, that

$$\int_{-1}^{1} P_n(x)\psi(x)\,dx = 0 \qquad (n = 0, 1, 2, \cdots).$$

According to our recursion formula (Sec. 73),

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x) \qquad (n = 1, 2, \cdots);$$

and this formula can be replaced by the formula $xP_0(x) = P_1(x)$ when $n = 0$. When we multiply its terms by $\psi(x)$ and integrate from -1 to 1, the integrals in the right-hand member vanish, so that

(2) $$dx = \qquad (n = 0, 1, 2, \cdots).$$

If we suppose that the orthogonality property (1) is true when $\psi(x)$ there is replaced by $x^k\psi(x)$, the method just employed clearly shows that property (1) is true when $\psi(x)$ is replaced by $x^{k+1}\psi(x)$. In view of equation (2), then, we conclude by induction that, for every integer j,

$$\int_{-1}^{1} P_n(x)x^j\psi(x)\,dx = 0 \qquad (n, j = 0, 1, 2, \cdots).$$

As a consequence, we have

$$dx = 0;$$

because the power series in the brackets, representing $\cos m\pi x$, is uniformly convergent. Moreover, the series obtained by multi-

plying all terms of this series by the sectionally continuous function $P_n(x)\psi(x)$ is also uniformly convergent, so it can be integrated termwise. Thus

$$\cos m\pi x \, dx = 0 \qquad (m = 0, 1, 2, \qquad);$$

and in just the same manner it follows that

$$\int_{-1}^{1} P_n(x)\psi(x) \sin m\pi x \, dx = 0 \qquad (m = 1, 2, \cdots).$$

All the coefficients in the Fourier series corresponding to the function $P_n(x)\psi(x)$ in the interval $(-1, 1)$ therefore vanish. But this function and its first derivative are sectionally continuous; hence it is represented by its Fourier series except at the points of discontinuity of the function. Except possibly at a finite number of points, then,

$$\psi(x) = 0 \qquad (-1 \leqq x \leqq 1),$$

and the theorem is proved.

There is an interesting consequence of the above theorem.

Suppose that for some real value of λ other than $n(n + 1)$, Legendre's equation

(3) $$\frac{}{dx}$$

has a solution $y = y_0(x)$, where $y_0'(x)$ is continuous in the interval $-1 \leqq x \leqq 1$. Then, according to Sec. 25, $y_0(x)$ is orthogonal to all the characteristic functions $P_n(x)$ already found and corresponding to $\lambda = n(n + 1)$. But this is impossible according to Theorem 1, unless $y_0 \equiv 0$.

Since we have already shown that λ must be real if equation (3) is to have such a regular solution, we have the following result:

Theorem 2. *The only values of λ for which the Legendre equation* (3) *can have a non-zero solution with a continuous derivative of the first order, in the interval $-1 \leqq x \leqq 1$, are*

$$\lambda = n(n + 1) \qquad (n = 0, 1, 2, \cdots).$$

It can be shown that the Legendre functions of the second kind, $Q_n(x)$, which also satisfy equation (3) when $\lambda = n(n + 1)$, become infinite at $x = \pm 1$. Consequently, *the polynomials $P_n(x)$*

are, except for constant factors, *the only solutions of equation* (3) *which have continuous first derivatives in the interval* $-1 \leqq x \leqq 1$.

77. The Expansion of x^m. Without the use of a general expansion theorem, we can easily show how every integral power of x, and therefore every polynomial, can be expanded in a finite series of the polynomials $P_n(x)$. It will be clear that these important expansions are valid for all x, not just for the values of x in the interval $(-1, 1)$.

According to its definition, the polynomial $P_m(x)$ has the form

$$P_m(x) = ax^m + bx^{m-2} + cx^{m-4} + \cdots, \tag{1}$$

where a, b, are constants depending on the integer m. Therefore

$$^{-2} - \frac{c}{a}; \quad ^{-4} -$$

That is, every integral power x^m of x can be written as a constant times $P_m(x)$ plus a polynomial in x of degree $m - 2$. Applying this rule to x^{m-2} in the last equation, we see that x^m is a linear combination of $P_m(x)$, $P_{m-2}(x)$, and a polynomial of degree $m - 4$. Continuing in this way, and noting that only the alternate exponents m, $m - 2$, $m - 4$, $\cdots$ appear in the polynomials here, it is clear that there is a finite series for x^m of the following form:

$$x^m = A_m P_m(x) + A_{m-2} P_{m-2}(x) + \cdots + T, \tag{2}$$

where the final term T is a constant A_0 if m is even, and

$$T = A_1 P_1(x)$$

if m is odd.

To find the value of any coefficient A_{m-2j}, we multiply all terms of equation (2) by $P_{m-2j}(x)$ and integrate over the interval $(-1, 1)$. In view of the orthogonality of the functions $P_n(x)$, this gives

$$dx = \qquad dx.$$

But the integrand on the left is an even function of x for every integer m; and the integral on the right, the norm of $P_{m-2j}(x)$, has the value $2/(2m - 4j + 1)$. Therefore,

$$_{-2j} = (2m - \qquad dx. \tag{3}$$

We shall develop here the following *integration formula, valid for every real number r greater than* -1:

$$(4) \quad \int_0^1 x^r P_n(x)\,dx = \frac{r(r-1)\cdots(r-n+2)}{(r+n+1)(r+n-1)\cdots(r-n+3)} \qquad (n = 2, 3, \cdots).$$

In view of this formula, the integral in equation (3) has the value

$$\frac{m(m-1)\cdots(2j+2)}{(2m-2j+1)(2m-2j- \quad\quad j+3)}$$

$$m!$$

$$1 \cdot 3 \cdot 5 \cdots (2m-2j$$

The values of the coefficients A_{m-2j} are therefore determined, and we can write expansion (2) for *any integral power of x* as follows:

$$1 \cdot 3 \cdot 5 \qquad \overline{(2m+1)} \quad (2m+1)P_m(x)$$

For the first few values of m, we have

$$1 = P_0(x), \qquad x = \qquad\qquad x^2 = \qquad\qquad + \tfrac{1}{3}P_0(x),$$

Derivation of Formula (4). To obtain the integration formula (4), let us first observe that, in view of expression (1),

$$(5) \quad \int_0^1 x^r P_n(x)\,dx$$

$$= \int_0^1$$

$$r+n+1 \quad r+n-1 \quad r+n-3$$

as long as $r > -1$. The last member can be written

$$(6) \qquad \frac{f(r)}{n-1)(r+n-3)},$$

where

$$f(r) = a(r + n - 1)(r + n - 3) \cdots + b(r + n + 1)(r + n - 3) \cdots + \cdots .$$

We can see that $f(r)$ is a polynomial in r of degree $n/2$ or $(n - 1)/2$, according as n is even or odd.

Now the product $x^{n-2j}P_n(x)$ is an even function of x for every n, when $j = 1, 2, \cdots$; so it is evident from equation (2) that

$$x^{n-2j}P_n(x)\,dx = 0 \qquad (j = 1, 2, \cdots ; n \geqq 2j).$$

Therefore our integral (5) vanishes when r is replaced by $n - 2$, $n - 4$, $n - 6$, etc., down to zero or unity, and so does the polynomial $f(r)$. Also, the coefficient of the highest power of r in $f(r)$ is $a + b + c + \cdots$, which is $P_n(1)$ or unity. Hence, when $n = 2, 3, \cdots$, the factors of $f(r)$ can be shown as follows:

$$f(r) = (r - n + 2)(r - n + 4) \cdots r,$$

if n is even; and

$$f(r) = (r - n + 2)(r - n + 4) \cdots (r - 1),$$

if n is odd. In either case the fraction (6) can be written as

$$\frac{r(r-1)(r-2) \cdots (r-n+2)}{(r+n+1)(r+n-1) \cdots (r-n+3)} \qquad (n = 2, 3, \cdots ; r > -1).$$

This is the value of our integral; hence formula (4) is established.

78. Derivatives of the Polynomials. The derivative $P_n'(x)$ is a polynomial of degree $n - 1$ containing alternate powers of x, namely, x^{n-1}, x^{n-3}, $\cdots$. It can therefore be written as a finite series of Legendre polynomials:

$$P_n'(x) = A_{n-1}P_{n-1}(x) + A_{n-3}P_{n-3}(x) + \cdots .$$

To find the coefficient A_j $(j = n - 1,\ n - 3, \cdots)$, we multiply all terms by $P_j(x)$ and integrate; thus

When integrated by parts, the integral here becomes

and this last integral vanishes because $P'_j(x)$ is a linear combination of the polynomials $P_{j-1}(x)$, $P_{j-3}(x)$, etc., each of which is of lower degree than $P_n(x)$. Therefore

$$- (-1)^{j+n}] = 2j$$

since $j + n = 2n - 1, 2n - 3, \cdots$.

Consequently we have the following expansion, valid for all x:

$$(1) \quad P'_n(x) = (2n - \qquad (2n -$$

ending with $3P_1(x)$ if n is even, and with $P_0(x)$ if n is odd.

When $-1 \leqq x \leqq 1$, we have seen that $|P_n(x)| \leqq 1$; hence for these values of x it follows from expansion (1) that

$$|P'_{2n}(x)| \leqq (4n - \qquad 4n - 5) \qquad + 3 = n(2n + 1);$$

and similarly,

$$1)(2n - 1).$$

Therefore,

that is, for all x in the interval $-1 \leqq x \leqq 1$,

$$(2) \qquad |P'_n(x)| \leqq n^2 \qquad (n = 0, 1, 2, \cdots).$$

Differentiating both members of expansion (1) and noting that $|P'_{n-1}(x)| \leqq n^2$, $|P'_{n-3}(x)| \leqq n^2$, etc., we see by the method used above that

$$(3) \qquad |P''_n(x)| \leqq n^4 \qquad (-1 \leqq x \leqq 1, n = 0, 1, 2, \cdots).$$

Similarly, for derivatives of higher order, $|P_n^{(k)}(x)| \leqq n^{2k}$.

Let us collect our properties on the order of magnitude of the Legendre coefficients and their derivatives as follows:

Theorem 3. *For all x in the interval $-1 \leqq x \leqq 1$, and for $n = 1, 2, 3, \cdots$, the values of the functions*

$$1, \ n^4$$

can never exceed unity.

79. An Expansion Theorem. The normalized Legendre polynomials were found in Sec. 75 to be

$$\varphi_n(x) = \sqrt{n + \tfrac{1}{2}}\, P_n(x) \qquad (n = 0, 1, 2, \cdots).$$

The Fourier constants, corresponding to the orthonormal set here, for a function $f(x)$ defined in the interval $(-1, 1)$, are

$$c_n = \int_{-1}^{1} f(x)\varphi_n(x)\, dx = \sqrt{n + \tfrac{1}{2}} \int_{-1}^{1} f(x) P_n(x)\, dx.$$

The generalized Fourier series corresponding to $f(x)$ is therefore

$$\cdot \frac{1}{} \int_{-1}^{1}$$

This can be written

where

$$(2) \qquad A_n = \frac{2n+1}{} \quad {}_{-1} f(x) P_n(x)\, dx \qquad (n = 0, 1, 2, \quad).$$

The series (1) with the coefficients (2) is called *Legendre's series* corresponding to the function $f(x)$. It was shown above that if $f(x)$ is any polynomial, this series contains only a finite number of terms and represents $f(x)$ for all values of x.

It can be shown that, when $-1 < x < 1$, Legendre's series converges to $f(x)$ under any of the conditions given earlier for the representation of this function by its Fourier series. We now state explicitly a fairly general theorem on such expansions, and accept it without proof for the purposes of the present volume.*

Theorem 4. *Let $f(x)$ be bounded and integrable in the interval $(-1, 1)$. Then at each point, x $(-1 < x < 1)$ which is interior to an interval in which $f(x)$ is of bounded variation, the Legendre series corresponding to $f(x)$ converges to $\tfrac{1}{2}[f(x + 0) + f(x - 0)]$; that is,*

$$(3) \qquad 0) \qquad - 0)] =$$

where the coefficients A_n are given by formula (2).

* The theorem stated here is a special case of a theorem proved in Chap. VII of Ref. 1. The proof is lengthy and involves more advanced concepts than we employ in this book.

In particular, *if $f(x)$ is sectionally continuous in the interval $(-1, 1)$, and if its derivative $f'(x)$ is sectionally continuous in every interval interior to* $(-1, 1)$, then expansion (3) is valid whenever $-1 < x < 1$. For it can be shown that the conditions in the theorem are then satisfied at all points.

If $f(x)$ is an *even function*, the product $f(x)P_n(x)$ is even or odd according as n is an even or odd integer. Hence $A_n = 0$ if n is odd, and

$$(4) \qquad A_n = (2n + 1) \int_0^1 f(x)P_n(x)\,dx \qquad (n = 0, 2, 4, \cdots);$$

so that expansion (3) becomes

(5)

where the coefficients are defined by formula (4).

Similarly, if $f(x)$ is an *odd function*, the expansion becomes

$$(6) \qquad x - 0)] \\ = A_1P_1(x)$$

where

$$(7) \qquad A_n = (2n + 1) \int_0^1 f(x)P_n(x)\,dx \qquad (n = 1, 3, 5,$$

In the interval $(0, 1)$, *either one of the expansions* (5) *or* (6) *can be used,* provided of course that $f(x)$ satisfies the conditions of the theorem in that interval. For if $f(x)$ is defined only in $(0, 1)$, it can be defined in $(-1, 0)$ so as to make it either even or odd in $(-1, 1)$. It was pointed out earlier (Sec. 75) that the polynomials $P_{2n}(x)$, and the polynomials $P_{2n-1}(x)$, appearing in expansions (5) and (6), respectively, form two sets of orthogonal functions on the interval $(0, 1)$.

When $x = \cos\theta$, expansion (3) can of course be written

$$F(\theta) = \sum_0^\infty A_nP_n(\cos\theta) \qquad (0 < \theta < \pi)$$

at points where $F(\theta)$ is continuous, where

$$A_n = \frac{2n + 1}{2} \int_0^\pi F(\theta)P_n(\cos\theta)\sin\theta\,d\theta \quad (n = 0, 1, 2, \cdots).$$

PROBLEMS

1. If $f(x) = 0$ when $-1 < x < 0$, $f(x) = 1$ when $0 < x < 1$, a $f(0) = \frac{1}{2}$, obtain the following expansion for $f(x)$ when $-1 < x < 1$:

$f(x) = \frac{1}{2} + \frac{1}{2}$

$4 \quad 2 \cdot 4$

Suggestion: See Prob. 4, Sec. 73.

2. If $f(x) = 0$ when $-1 < x \leqq 0$, and $f(x) = x$ when $0 \leqq x < 1$, show that, when $-1 < x < 1$,

$\frac{1}{2^2}$

$13 \cdot 4!$

3. Expand the function $f(x) = x$, when $0 \leqq x < 1$, in series of Legendre polynomials of even order, in the interval (0, 1).

80. The Potential about a Spherical Surface. Let a spherical surface be kept at a fixed distribution of electric potential $V = F(\theta)$, where r, φ, θ are spherical coordinates with the origin at the center of the sphere. The potential at all points in the space, assumed to be free of charges, interior to and exterior to the surface is to be determined. It will clearly be independent of φ; hence it must satisfy the following case of Laplace's equation in spherical coordinates:

The potential $V(r, \theta)$ will also be required to be continuous, together with its second-order derivatives, in every region not containing a point of the surface, and to vanish at points infinitely far from the surface. The boundary conditions are therefore

$$\lim_{r \to c} V(r, \theta) = F(\theta) \qquad (0 < \theta < \pi), \tag{2}$$

where c is the radius of the spherical surface, and

$$\lim_{r \to \infty} V(r, \theta) = 0. \tag{3}$$

Particular solutions of equation (1) can be found by the usual method. Setting $V = R(r)\Theta(\theta)$, equation (1) becomes

Both members here must be equal to a constant, say λ; hence we have the equations

$$\frac{d^2}{dr^2}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1-\cos^2\theta}{\sin\theta}\frac{d\Theta}{d\theta}\right.$$

The first of these is Cauchy's linear equation,

$$r^2R'' + 2rR' - \lambda R = 0,$$

which can be reduced to one with constant coefficients by substituting $r = e^t$. Its general solution is

Writing $-\frac{1}{2} + \sqrt{\lambda + \frac{1}{4}} = n$, so that $\lambda = n(n+1)$, we have

$$R(r) = Ar^n + \frac{B}{r^{n+1}},$$

where n is any constant.

Writing x for $\cos\theta$, the equation in Θ becomes, in terms of the new parameter n,

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + n(n+1)\Theta = 0,$$

which is *Legendre's* equation. We have seen that the solution of this equation can be continuous, together with its first ordered derivative, in the interval $-1 \leqq x \leqq 1$, or $0 \leqq \theta \leqq \pi$, only if n is an integer. The solutions are then the Legendre polynomials, which have continuous derivatives of all orders. Hence

$$n = 0, 1, 2, \cdots,$$

and

$$\Theta = P_n(x) = P_n(\cos\theta).$$

Thus two sets of particular solutions $R\Theta$ of Laplace's equation (1) have been determined:

$$(4)\qquad r^nP_n(\cos\theta);\qquad r^{-n-1}P_n(\cos\theta)\qquad (n = 0, 1, 2, \cdots).$$

In the first set the functions and their derivatives of all orders with respect to r or θ are continuous in every finite region; and in the second set they are continuous in every region, finite or infinite, not containing the origin.

Then at points inside the sphere the function

$$V(r, \theta) = \qquad (r < c)$$

satisfies (1) and (2) formally, provided B_n can be determined so that

$$f(\cos\theta) = \sum_0^\infty B_n c^n P_n(\cos\theta) \qquad (0 < \theta < \pi),$$

where $f(\cos\theta) = F(\theta)$. This is the expansion of the last section, provided $B_n c^n$ are the coefficients A_n given there; that is, if

$$1 \; 2n \quad \frac{1}{} \int_0^\pi f(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta.$$

Hence *for points inside* the sphere, the solution of the problem can be written

$$(5) \qquad V(r, \theta) = \qquad \frac{n+1}{2} \frac{r^n}{c^n} P_n(\cos\theta) \int_{-1}^{1} f(x) P_n(x)\, dx$$
$$[r \leqq c;\ F(\theta) = f(\cos\theta)].$$

For points exterior to the sphere, the functions of the second set in (4) satisfy condition (3), and the solution can be written

$$(6) \qquad V(r, \theta) = \sum A_n \frac{c^{\cdots}}{r^{n+1}}. \qquad (r \geqq c),$$

where

(7)

since the series in (6) then reduces to $f(\cos\theta)$ when $r = c$.

The Solution Established. To show that our formal solution does satisfy all the conditions of the problem, we use the same method here as in earlier problems (for example, Sec. 46). We shall suppose that the given function $F(\theta)$ and its derivative $F'(\theta)$ are sectionally continuous in the interval $(0, \pi)$. Then

$f(x)$ is sectionally continuous in the interval $(-1, 1)$, and so is $f'(x)$, in every interval interior to $(-1, 1)$.

Now consider the function $V(r, \theta)$ represented by formula (5). When $r = c$, the series there converges to $f(x)$ if $-1 < x < 1$. But the sequence of functions $(r/c)^n$ $(n = 0, 1, 2, \cdots)$ is bounded, and monotone with respect to n; hence according to Abel's test the series is uniformly convergent with respect to r $(0 \leqq r \leqq c)$ for each fixed x $(-1 < x < 1)$. Therefore $V(c - 0, \theta) = V(c, \theta)$, and so condition (2) is satisfied.

The terms of the series in equation (5) can be written as the product of the three factors A_n/n, $P_n(\cos \theta)$, and $n(r/c)^n$. Since the first two factors are bounded for all r, θ, and n

and since the series whose general term is the last factor converges when $r < c$, the series in equation (5) is uniformly convergent when $r < c$. But the series of terms $n^k(r/c)^n$, for each fixed positive k, also converges when $r < c$; and since $n^{-2}P_n'(x)$ and $n^{-4}P_n''(x)$ are uniformly bounded (Theorem 3), it follows easily that the series in equation (5) can be differentiated termwise twice with respect to r and with respect to θ, when $r < c$. The individual terms of that series satisfy Laplace's equation (1); hence our function $V(r, \theta)$ satisfies that equation. Also, $V(r, \theta)$ and its derivatives are continuous when $r < c$.

This establishes our solution when $r < c$. When $r > c$, solution (6) can be proved valid in the same manner. If, as a periodic function of the angle θ, $F(\theta)$ is supposed continuous and $F'(\theta)$ sectionally continuous, it is also possible to show that the above solutions are the only possible solutions satisfying certain regularity conditions, essentially that $V(r, \theta)$ be continuous at $r = c$ (see Sec. 58).

PROBLEMS

1. If the potential is a constant V_0 on the spherical surface of radius c, show that $V = V_0$ at all interior points, and $V = V_0c/r$ at each exterior point.

2. Find the steady temperatures at points within a solid sphere of unit radius if one hemisphere of its surface is kept at temperature zero and the other at temperature unity; that is, $f(\cos \theta) = 0$ when $\pi/2 < \theta < \pi$, and $f(\cos \theta) = 1$ when $0 < \theta < \pi/2$.

Ans. $u(r, \theta) = \frac{1}{2} + \frac{3}{4}\, r \cos \theta - \frac{7}{8}\, \frac{1}{2} r^3 P_3(\cos \theta)$

$\theta) - \cdots .$

3. Find the steady temperatures $u(r, \theta)$ in a solid sphere of unit radius if $u = C \cos \theta$ on the surface. *Ans.* $u = Cr \cos \theta$.

4. Find the potential V in the infinite region $r > c$, $0 \leqq \theta \leqq \pi/2$, if $V = 0$ on the plane portion of the boundary ($\theta = \pi/2$, $r > c$) and at $r = \infty$, and $V = f(\cos \theta)$ on the hemispherical portion of the boundary ($r = c$, $0 \leqq \theta \leqq \pi/2$).

$$V(r, \theta) = \sum_{0}^{\infty} (4n + 3)(c/r)^{2n+2} P_{2n+1}(\cos \theta) \int_0^1 f(x) P_{2n+1}(x)\, dx.$$

5. Find the steady temperatures $u(r, \theta)$ in a solid hemisphere of radius c whose convex surface is kept at temperature $u = f(\cos \theta)$, if the base is insulated; that is,

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 \qquad \text{when } \theta = \frac{\pi}{2}.$$

Also write the result when $f(\cos \theta) = 1$.

$$u(r, \theta) \qquad (4n \qquad\qquad \theta) \qquad\qquad dx.$$

6. Find the steady temperatures in a solid hemisphere of unit radius if its convex surface is kept at temperature unity and its base at temperature zero.

7. Show that the steady temperature $u(r, \theta)$ in a hollow sphere with its inner surface $r = a$ kept at temperature $u = f(\cos \theta)$, and its outer surface $r = b$ at $u = 0$, is

$$u = \sum_{0} A_n \frac{b^{2n+1} - r^{2n+1}}{b^{2n+1} - a^{2n+1}} \left(\frac{a}{r}\right.$$

where

$$2n \qquad\qquad (x) P_n(x)\, dx.$$

8. If $u(x, t)$ represents the temperature in a nonhomogeneous bar with ends at $x = -1$ and $x = 1$, in which the thermal conductivity is proportional to $1 - x^2$, and if the lateral surface of the bar is insulated, the heat equation has the form

where b is a constant, provided the thermal coefficient $c\delta$ is constant (Sec. 9). The ends $x = \pm 1$ are also insulated because the conductivity vanishes there. If $u = f(x)$ when $t = 0$, derive the following formula

for $u(x, t)$:

$$u = \sum^{\infty}$$

9. When the initial temperature function in Prob. 8 is (*a*) $f(x) = x^2$, (*b*) $f(x) = x^3$, show that the solution reduces to the following formulas, respectively:

(*a*) $$u =$$

81. The Gravitational Potential Due to a Circular Plate. Another type of application of Legendre polynomials to the solution of boundary value problems will be illustrated by the following problem:

Find the gravitational potential due to a thin homogeneous circular plate, or disk, of mass δ per unit area and radius c.

Let the center of the disk be taken as the origin and the axis as the z-axis, $\theta = 0$, where r, φ, θ are spherical coordinates. The potential is a function $V(r, \theta)$ independent of φ; hence it satisfies the following form of Laplace's equation:

$$\partial V$$

except at points in the disk. Its value at points on the positive axis $\theta = 0$ can be found from the definition of potential by a simple integration; thus

$$V(r, 0) =$$

where k is the gravitational constant in the definition of potential. Then $V(r, \theta)$ must be symmetric with respect to the origin and satisfy the following boundary condition in the space $0 \leqq \theta < \pi/2$, $r > 0$:

(2) $$V(r, 0) =$$

where M is the mass of the disk.

Two solutions of equation (1) were found in the last section, namely,

$$V = \sum_0 a_n r^n P_n(\cos\theta), \tag{3}$$

(4)

The coefficients a_n, b_n are now to be determined, if possible, so that boundary condition (2) is satisfied. But when $\theta = 0$, $P_n(\cos\theta) = 1$ and the series in (3) and (4) become power series in r and in reciprocals of r, respectively.

Now the binomial expansions

$$= c\Big(1 \qquad \frac{1}{2 \cdot 4}\frac{r^4}{c^4} + \frac{\quad}{2 \cdot 4 \cdot 6}\frac{\quad}{c^6} \qquad \cdots\Big) \qquad (0 \leqq r < c),$$

$$= r\Big(1 \qquad \frac{1}{2 \cdot 4}\frac{c}{r^4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{\quad}{r^6} \qquad (r > c),$$

are absolutely convergent in the indicated intervals, and convergent when $r = c$. Hence boundary condition (2) can be written

$$V(r, 0) = \frac{2Mk}{\quad} \qquad \frac{1}{2 \cdot 4}\frac{r^4}{c^4} \tag{5}$$

$$\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{r^6}{c^6} \qquad \cdots\Big) \quad \text{when } 0 < r \leqq c;$$

$$\frac{\quad}{2r} - \frac{\quad}{2 \cdot 4\, r^3} \quad \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{c^5}{r^5} - \cdots\Big) \quad \text{when } r \geqq c.$$

The series in (3) will then satisfy (5) for $r < c$ if its coefficients are identified with those of the first series in (5); thus $a_0 = 2Mk/c$, $a_1 = -2Mk/c^2$, etc. Similarly, for the case $r > c$ the series in (4) can be used if its coefficients are taken as those in the second series in (5), namely, $b_0 = Mk$, $b_1 = 0$, etc.

Hence the solution of the boundary value problem (1)-(2) can be written as follows, when $0 \leqq \theta < \pi/2$:

$$V(r, \theta) \qquad \text{os } \theta) + \frac{1}{2}\frac{r^2}{c^2} P_2(\cos\theta)$$

$$\theta) + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{r^6}{c^6} P_6(\cos\theta) -$$

if $0 < r < c$; and

$$+ \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \frac{c^5}{r^5} P_4(\cos\theta) - \cdots \Bigg],$$

if $r > c$.

When $r \neq c$, the convergence of the series here follows from the absolute convergence of the series in (5) and the fact that $|P_n(\cos\theta)| \leqq 1$.

PROBLEMS

1. Derive the following formula for the gravitational potential due to a mass M distributed uniformly over the circumference of a circle of radius 1, when $0 \leqq \theta \leqq \pi$:

$$V(r, \theta) = kM \left[1 - \frac{1}{2} r^2 P_2(\cos\theta) + \frac{1 \cdot 3}{2 \cdot 4} \quad \theta) - \quad \right],$$

if $0 \leqq r < 1$; and

$$V(r, \theta) \quad kM \left[\frac{1}{r} - \frac{1}{2} \quad - \cdots \right],$$

if $r > 1$.

2. Find the gravitational potential, at external points, due to a solid sphere, taking the unit of mass as the mass of the sphere, and the unit of length as the radius, if the density of the sphere is numerically equal to the distance from the diametral plane $\theta = \pi/2$.

Ans. $$V(r, \theta) = k \left[\frac{1}{r} + \frac{1}{6} \frac{P_2(\cos\theta)}{r^3} - \frac{1}{6 \cdot 8} \frac{P_4(\cos\theta)}{r^5} + \frac{1 \cdot 3}{6 \cdot 8 \cdot 10} \frac{P_6(\cos\theta)}{r^7} - \cdot \cdot \right.$$

3. Find the gravitational potential, at external points, due to a hollow sphere of mass M and radii a and b, if the density is proportional to the distance from the diametral plane $\theta = \pi/2$.

4. The points along the z-axis, $\theta = 0$ or $\theta = \pi$, in an infinite solid are kept at temperature $u = Ce^{-r^2}$. Find the steady temperature $u(r, \theta)$ at all points. *Ans.* $u(r, \theta) = C \sum_0^\infty (-1)^n (r^{2n}/n!) P_{2n}(\cos\theta)$.

5. The surface $\theta = \pi/3$, $r \geqq 0$, of an infinitely long solid cone is kept at temperature $u = Ce^{-r}$. Find the steady temperatures $u(r, \theta)$ in the cone. *Ans.* $u(r, \theta) \quad C \sum_0 (-1)^n [r^n P_n(\cos$

6. Solve Prob. 5 if the surface temperature is $u(r, \pi/3) = C/r^m$, where m is a fixed positive integer.

REFERENCES

1. Hobson, E. W.: "The Theory of Spherical and Ellipsoidal Harmonics," 1931.
2. Whittaker, E. T., and G. N. Watson: "Modern Analysis," 1927.
3. Byerly, W. E.: "Fourier's Series and Spherical Harmonics," 1893. Appendix.

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